MA 331 Extreme Value Theorem

Lemma 1. If X is a Hausdorff space and if $P \subset X$ is compact, then P is closed.

Lemma 2. Suppose that X is compact and that for each $n \in \mathbb{N}$, $K_n \subset X$ is closed and non-empty. Furthermore, assume that for all n, $K_{n+1} \subset K_n$. Then

$$\bigcap_{n \in \mathbb{N}} K_n \neq \varnothing.$$

Theorem 3 (Extreme Value Theorem)**.** Let K be a non-empty compact topological space and let $f\colon K \to \mathbb{R}$ be continuous. Then there exist $m, M \in K$ such that for all $x \in K$, we have

$$f(m) \leq f(x) \leq f(M)$$

Proof. Let $V_t = (-\infty, t) \subset \mathbb{R}$ and let $U_t = f^{-1}(V_t)$. Since V_t is an open interval and since f is continuous, each U_t is open. Let $K_t = U_t^C$. Suppose, first, that for all $t \in \mathbb{R}$, $K_t \neq \varnothing$. Then by Lemma 2, there is an $x \in \bigcap_{n \in \mathbb{N}} K_n$. However that means that for all $n \in \mathbb{N}$, the value $f(x) \notin (-\infty, n)$. This contradicts the fact that $\mathbb{R} = \bigcup_{n \in \mathbb{N}} (-\infty, n)$. Thus, there is a t such that $K_t = \varnothing$. Let

$$t_0 = \inf\{t : K_t = \varnothing\}.$$

Since $K \neq \varnothing$, $t_0 \neq -\infty$, that is $t_0 \in \mathbb{R}$.

We claim that there exists $M \in K$ such that $f(M) = t_0$. Suppose, to the contrary, that there is no such M. Then $t_0 \notin f(K) \subset \mathbb{R}$. Since K is compact, $f(K)$ is compact. Since $\mathbb{R}$ is Hausdorff, $f(K)$ is closed (Lemma 1). By the definition of closed, there exists $\varepsilon > 0$ such that the interval $(t_0 - \varepsilon, t_0 + \varepsilon) \not\subset f(K)$. In particular, the set $K_{t_0 - \varepsilon/2} = \varnothing$. This contradicts the choice of t_0 and so there must be such an $M \in K$.

Notice that if $s > t_0$, then $K_s = \varnothing$.

We now show that M is a global maximum for f. Let $x \in K$. If $f(x) > t_0 = f(M)$ then $x \in K_{t_0}$. Letting $\varepsilon = (f(x) - t_0)/2$, we see that also $x \in K_{t_0 + \varepsilon}$. This contradicts the fact of the previous paragraph. Thus, $f(x) \leq t_0 = f(M)$ as desired.

Finally we prove the existence of a global minimum. Let $g\colon K \to \mathbb{R}$ be the function where, for all $x \in K$, $g(x) = -f(x)$. It is easily seen that g is continuous. By our work above, g has a global maximum M_g such that for all $x \in K$, we have $g(x) \leq M_g$. Letting $m = -M_g$, we see that for all $x \in K$, $f(x) \geq m$, as desired. $\square$

Here is a somewhat shorter proof:

Proof. Let $f\colon X \to \mathbb{R}$ with X compact. Then $f(X) \subset \mathbb{R}$ is compact. Since $\mathbb{R}$ is Hausdorff, $f(X)$ is closed. Thus, $a = \inf f(X) \in f(X)$ and $b = \sum f(X) \in f(X)$.

Consequently, there is $m \in X$ with $f(m) = a$ and $M \in X$ with $f(M) = b$ and

$$f(m) \leq f(x) \leq f(M)$$

for all $x \in X$. $\square$