MA 122: Weekly HW 1

Answer these questions on a separate sheet of paper. Remember that your work must be very neat and complete.

Problem 1: Let $f(x) = \cos(x)$.

(a) Find a formula for the $2n$th MacLaurin Polynomial $P_{2n}(x)$ of the function $f(x)$. (Recall that a MacLaurin polynomial is simply a Taylor polynomial based at zero.) Why are we only considering the even numbered polynomials?

(b) Using Mathematica, graph $f(x)$ and $P_2(x)$ on the same plot. Make 2 more plots to compare $f(x)$ to $P_4(x)$, and $P_6(x)$. For your plot use x values between 0 and π. Write a paragraph comparing the accuracy of P_2, P_4, and P_6 as approximations to f. Where are they good approximations? Where are they bad approximations? Is one of them always a better approximation to $f(x)$ than the other (no matter what x is)?

(c) Use Taylor's theorem to find an upper bound for the error term $E_{2n}(x)$ for $x \in \mathbb{R}$.

Problem 2: The hyperbolic cosine function is defined to be

$$\cosh(x) = \frac{e^x + e^{-x}}{2}.$$

The hyperbolic sine function is defined to be

$$\sinh(x) = \frac{e^x - e^{-x}}{2}.$$

(a) Find a formula for the nth derivative of $\cosh(x)$ in terms of $\cosh(x)$ and $\sinh(x)$. (Hint: Your answer will depend on whether n is even or odd.)

(b) Find a formula for the $2n$th MacLaurin Polynomial $P_{2n}(x)$ of the function $\cosh(x)$. Why are we only considering the even numbered polynomials?

(b) Using Mathematica, graph $f(x)$ and $P_2(x)$ on the same plot. Make 2 more plots to compare $f(x)$ to $P_4(x)$, and $P_6(x)$. Write a paragraph comparing the accuracy of P_2, P_4, and P_6 as approximations to cosh. Where are they good approximations? Where are they bad approximations? Is one of them always a better approximation to $f(x)$ than the other (no matter what x is)?

(c) Use Taylor's theorem to find an upper bound for the error term $E_{2n}(x)$ for $x \in [0,2]$.

Problem 3: Let $f(x) = \sqrt[3]{x}$.

(a) Find a formulas for the 1st, 2nd, 3rd, 4th, and 5th Taylor Polynomials for $f(x)$ based at $x = 1$.

(b) Using Mathematica, graph $f(x)$ and $P_2(x)$ on the same plot. Make 3 more plots to compare $f(x)$ to $P_3(x)$, $P_4(x)$, and $P_5(x)$. Make your plots for x between 1 and 4.

Write a paragraph comparing the accuracy of P_2, P_3, P_4, and P_5 as approximations to $f(x)$. Where are they good approximations? Where are they bad approximations? Is one of them always a better approximation to $f(x)$ than the other (no matter what x is)?

Problem 4: In class we proved Taylor's Theorem. This problem is intended to make you think about the proof, by repeating and adapting certain aspects of it. Throughout you may assume that Taylor polynomials are based at zero and that $f(x)$ is infinitely differentiable.

(a) Suppose that $f(x)$ is a function such that $f''(x) \leq 8$ for all $x \in \mathbb{R}$. Mimic the proof of Taylor's theorem to show that

$$f(x) - P_1(x) \leq 4x^2$$

when $x \geq 0$.

(b) Suppose that $f(x)$ is a function such that $f^{(4)}(x) \leq 8$ for all $x \in \mathbb{R}$. Mimic the proof of Taylor's theorem to find an upper bound for $f(x) - P_3(x)$ when $x \geq 0$.

(c) Suppose that $f(x)$ is a function such that $M \leq f^{(4)}(x)$ for all $x \in \mathbb{R}$ and some fixed $M \in \mathbb{R}$. Adapt the proof of Taylor's theorem to show that

$$\frac{M}{24}x^4 \leq f(x) - P_3(x)$$

whenever $x \geq 0$.

Problem 5: Suppose that $f(x)$ is an infinitely differentiable function and that for all $n \geq 1$ and all $x \in \mathbb{R}$, $|f^{(n)}(x)| \leq (n-1)!$.

(a) Use Taylor's theorem to find an upper bound on $E_n(x) = |f(x) - P_n(x)|$ where $P_n(x)$ is the nth MacLaurin polynomial for $f(x)$.

(b) If $|x| \leq 1$, what happens to $E_n(x)$ as $n \to \infty$? What does this say about the MacLaurin Polynomials $P_n(x)$?