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Boring Split Links and Unknots

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requirement for the degree of Doctor of Philosophy in Mathematics

by

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Dedication

For

Stephanie		*George*
who moved to	*and*	*who was born*
California		*here*

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I am very grateful for Martin Scharlemann, my advisor, who suggested that I learn sutured manifold theory. I am also thankful for the helpful suggestions he made regarding the mathematics in and the writing of this thesis.

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> *The solution of a geometry problem does not in itself constitute a precious gift, but ... it is the image of something precious. Being a little fragment of particular truth, it is a pure image of the unique, eternal, and living Truth, the very Truth that once in a human voice declared: "I am the Truth." Every school exercise, thought of in this way, is like a sacrament.*

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Abstract

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by

Scott Allen Taylor

Boring is an operation that converts a knot or two-component link L_α in a 3–manifold M into another knot or two-component link L_β. It generalizes many classical operations in knot theory, such as rational tangle replacement and the Kirby band move. It is particularly interesting to ask about the properties of L_β if L_α is a split link or unknot. Boring is the knot-theory version of an operation, called "refilling a meridian", on a 3–manifold M containing a genus two handlebody W. Refilling a meridian is, in turn, an example of the well-known operation of adding a 2–handle to the boundary of a 3–manifold. This dissertation develops sutured manifold techniques which are useful for studying essential surfaces in 3–manifolds obtained by adding a 2–handle to the boundary of a 3–manifold. Some of the main results include criteria guaranteeing that a knot or link L_β obtained by boring a split link is hyperbolic, a solution for a large class of pairs (M,W) of a conjecture of Scharlemann concerning refilling meridians of a genus two handlebody, and criteria guaranteeing that adding a 2–handle to a genus two boundary component of a simple 3–manifold produces a simple 3–manifold.

These results also give new proofs of classical theorems concerning rational tangle replacement and Seifert surfaces of tunnel number one knots and links. For example, new proofs are given of the fact that composite knots have unknotting number greater than one, that genus is super-additive under band connect sum, and that tunnel number one knots and links have a minimal genus Seifert surface disjoint from a given tunnel.

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Notation and Conventions

M	usually a compact, orientable 3–manifold containing a genus 2–handlebody
W	a genus 2 handlebody
N	a compact, orientable 3–manifold; often $M - \mathring{W}$
F	a boundary component, usually of genus 2, of N; often the boundary of W
a	an essential curve in F
α	the core of a 2–handle attached to F along a
$\overline{\alpha}$	the cocore of the 2–handle with core α
b	an essential curve in F
β	usually the core of a 2–handle attached to b; occasionally a 1–complex in a sutured manifold
$\overline{\beta}$	the cocore of the 2–handle with core β
b^*	if b is non-separating, a curve bounding a thrice-punctured sphere with $\partial\eta(b) \subset F$
β^*	the core of a 2–handle attached to b^*

$N[a]$	the 3–manifold obtained by attaching a 2–handle to the curve a
$M[\alpha]$	the 3–manifold obtained by refilling the meridian α of W
∂X	the boundary of a space X
$\partial_1 N[a]$	$\partial N - F$
$\partial_0 N[a]$	$\partial N[a] - \partial_1 N$
γ	sutures on $\partial N[a]$
$\widehat{\gamma}$	sutures on $\partial_0 N[a]$; that is $\gamma \cap \partial_0 N[a]$
$\eta(X)$	a closed regular neighborhood of a space X
$\mathrm{cl}(X)$	the closure of a space X
$\mathring{X}$, $\mathrm{int} X$	the interior of the space X
$\lvert X \rvert$	the number of connected components of X
Q	usually a parameterizing surface in a sutured manifold
$\overline{Q}$	usually a surface in $N[b]$ whose intersection with N is a parameterizing surface.
q	usually the number of components of ∂Q parallel to b
q^*	usually the number of components of ∂Q parallel to b^*
$\widetilde{q}$	$q + q^*$
S^n	the n–dimensional sphere.
D^2	the 2–dimensional unit disc
I	the unit interval
$\mathbb{N}$	the natural numbers

We work in the smooth or PL category; in particular, all surfaces in 3–manifolds are tame. All homology and cohomology groups have integer coefficients.

CHAPTER 1

Boring Knots and Refilling Meridians

1.1. Boring and Genus 2 Handlebodies

Given a genus two handlebody W embedded in a 3–manifold M, a knot or two-component link can be created by choosing an essential disc $\alpha \subset W$ and boundary-reducing W along α. That is, $W - \mathring{\eta}(\alpha)$ is the regular neighborhood of a knot or link L_α. We say that the exterior $M[\alpha]$ of this regular neighborhood is obtained by **refilling** the meridian disc α [**S5**]. Similarly, given a knot or link $L_\alpha \subset M$ we can obtain another knot or link L_β by the following process:

(1) Attach an arc to L_α forming a graph
(2) Thicken the graph to form a genus two handlebody W.
(3) Choose a meridian β for W and refill β.

The arc in step (1) and the handlebody in step (2) are called the **boring arc** and the **boring handlebody** respectively. Refilling the meridian α of the added arc returns L_α. Any two knots in S^3 can be related by such a move if we allow α and β to be disjoint; just let W be a neighborhood of the wedge of the two knots. We'll restrict attention, therefore, to meridians of W which cannot be isotoped to be disjoint. If a knot or link L_β can be obtained from L_α by this operation say that L_β is obtained by **boring** L_α.

Since the relation is symmetric we may also say that L_α and L_β are **related by boring**.

EXAMPLE. Every tunnel number 1 knot or link in S^3 can be obtained by boring a split link or unknot using an unknotted genus 2 handlebody W (i.e. a handlebody which is half of a genus 2 Heegaard splitting for S^3). Figure 1.1 shows how the trefoil knot can be obtained from the unknot by boring.

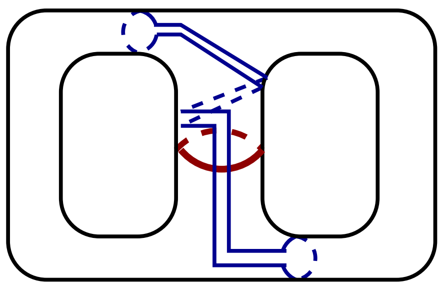

FIGURE 1.1. Cutting along one of the pictured discs produces an unknot; cutting along the other produces a trefoil knot.

Boring generalizes several well-known operations in knot theory. These include rational tangle replacements such as band sums and crossing changes and the Kirby band move [**K1, FR**]. Figures 1.2 and 1.3 show an example of how a Kirby band move can be accomplished by boring. The band move begins with a framed oriented link and creates another framed link by attaching a band which joints one component to a push-off of the other component. (In the figures, the framing of the knot on the right is ± 3, depending on orientations.)

Figure 1.4 shows how a rational tangle replacement can be accomplished with boring. One disc has boundary the equator of the ball. The other disc is formed by banding two ends of the attached one-handles together by a

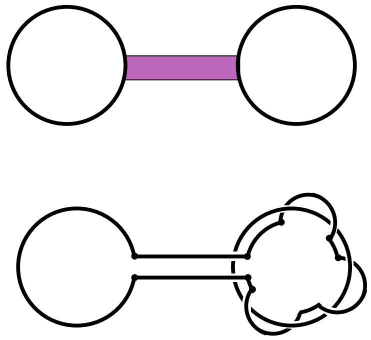

FIGURE 1.2. An example of a Kirby band move.

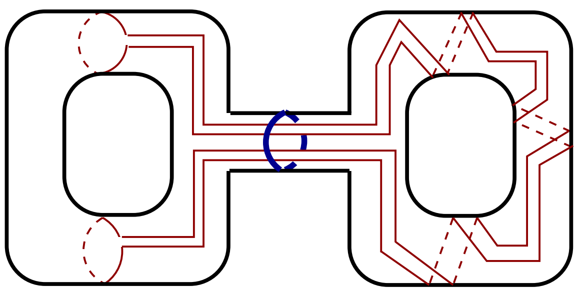

FIGURE 1.3. The Kirby band move from Figure 1.2 as boring. Cutting along one of the pictured disc produces the original link; cutting along the other disc produces the link after the band move.

band which can be isotoped into the four-punctured sphere. In the figure, only the core of the band is drawn.

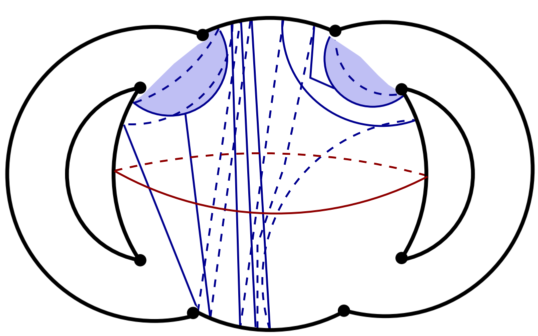

FIGURE 1.4. A rational tangle replacement operation is boring.

By work of Bleiler, Eudave-Muñoz, Scharlemann, and others, rational tangle replacements producing split links and unknots are fairly well understood; this understanding motivates many of the results contained in this thesis. Section 1.4 describes rational tangle replacements and their connection to the operations of boring and refilling. Many classical theorems are reproved from this new perspective in Section 8.4. New information about essential surfaces in the exterior of such a knot or link is also obtained. Sections 8.2 and 8.3 carry out this study.

Shifting away from a knot-theoretic viewpoint to a 3-manifold-theoretic viewpoint, we can view the operation of refilling meridians as a special case of 2–handle attachment. The main question under consideration is, "Given an essential curve in the boundary of a 3–manifold, what conditions guarantee that attaching a 2–handle to that curve produces a 3–manifold which is irreducible and boundary-irreducible?" Typically this question is answered by placing elementary conditions on the original 3–manifold (e.g. irreducible, boundary-irreducible, simple) and then bounding the intersection number between any two curves which produce reducible or boundary-reducible 3–manifolds.

This thesis proves two new theorems about attaching 2–handles to non-separating curves on a genus two boundary component. These are described in the next section.

1.2. 2–handle addition

Let N be an orientable 3-manifold and $F \neq S^2$ a non-empty boundary component. If $a \subset F$ is an essential simple closed curve, we can form a new 3-manifold $N[a]$ by attaching a 2–handle to a. Let $H = \alpha \times I$ where α is a 2–disc and let $f\colon\ \partial\alpha \times I \to \eta(a)$ be a homeomorphism such that $f(\partial\alpha) = a$. If F is not a torus, $N[a]$ is defined to be $N \cup_f H$. If F is a torus, $N \cup_f H$ has an additional spherical boundary component which was obtained by cutting F along a and attaching $\alpha \times \partial I$ to the boundary of F. Form $N[a]$ by gluing a 3–ball to this spherical boundary component. When F is a torus, attaching a 2–handle to N along a in F is more conventionally known as **Dehn-filling** N with slope a in F. For a genus 2 handlebody W embedded in a 3–manifold M, refilling a meridian α of W is equivalent to attaching a 2–handle to $M - \mathring{W}$ along $\partial\alpha$.

A fundamental result of Jaco [**J**] (generalizing a result of Przytycki) says that if F is compressible in N but $F - a$ is incompressible in N then $N[a]$ has incompressible boundary. Attempts to extend this result usually attempt to to compare the manifolds obtained by attaching a 2–handle to a curve $a \subset F$ and by attaching a 2–handle to a curve $b \subset F$ where a and b are curves that cannot be isotoped to be disjoint. The goal is then to conclude something about the geometry of the curves a and b based on the structures of $N[a]$ and $N[b]$. Much is known about the case when F is a torus. For example, if N is a knot exterior in S^3 and a is a meridian of F then if $N[b]$ is reducible a and b intersect exactly once [**GLu1**]. (The Cabling Conjecture asserts that, in fact, the knot is a cable knot and the surgery slope the slope of the cabling annulus. This is discussed more in Section 8.3). Other Dehn-surgery results (e.g. [**GW**]) include (often sharp) upperbounds on the minimal intersection number $\Delta(a,b)$ of a and b if N is hyperbolic but neither $N[a]$ nor $N[b]$ is hyperbolic.

When F is not a torus, far less is known. Still, there are some important results. Scharlemann and Wu [**SW**], for example, prove that if N is hyperbolic then if $N[a]$ is reducible and $N[b]$ is boundary-reducible, either a and b can be isotoped into a common once-punctured torus or $\Delta = 0$. More recently, Zhang, Qiu, and Li [**ZQL**] have shown that if N is hyperbolic, if a and b are separating curves, and $N[a]$ and $N[b]$ are reducible then $\Delta \leq 4$. They have also shown [**LQZ**] that if N is hyperbolic and F has genus 2 then there is at most one separating slope a so that $N[a]$ is boundary-reducible. (In this paper, hyperbolic will always mean the same thing as **simple**. That is, N is **simple** if it is irreducible, boundary-irreducible, anannular, and atoroidal. Since we are always studying compact orientable 3–manifolds with non-spherical boundary, by Thurston's geometrization theorem this is equivalent to having a finite volume hyperbolic structure on the manifold obtained by removing torus boundary components. We do not use this fact.)

Here are two new results concerning 2–handle addition.

THEOREM 6.1. *Suppose that F has genus 2, that N is compact, orientable, and irreducible, that $\partial N - F$ consists of tori, that N is boundary-irreducible, and that there is no essential annulus in N with both boundary components parallel to $a \subset F$ or both boundary components parallel to $b \subset F$. If a and b*

are non-isotopic separating non-parallel curves, then one of $N[a]$ and $N[b]$ is irreducible.

THEOREM 6.2. *Suppose that F has genus 2, and that N is simple. Suppose that a and b are non-isotopic separating curves on F. Suppose that $N[a]$ is reducible. Then if $N[b]$ is non-simple it contains an essential annulus with boundary on non-torus components of $\partial N[b]$ and $\Delta = 4$.*

1.3. Scharlemann's Conjecture

The remaining results concern the situation of refilling meridians α and β of a genus 2 handlebody W embedded in a 3–manifold M. Scharlemann, in the paper [**S5**] which introduces this idea, formulated a conjecture about circumstances guaranteeing that $M[\alpha]$ or $M[\beta]$ would be irreducible and boundary-irreducible. He proved the conjecture, or closely related statements, in several situations, most prominently when ∂W compresses in $N = M - \mathring{W}$ or when at least one of α or β is non-separating. He suggests that sutured manifold theory might aid in the complete resolution of the conjecture.

In this paper, a solution using sutured manifold theory is given for a number of 3–manifolds M and a number of embeddings of W in M. Here is the result which is most easily stated. More detail on Scharlemann's conjecture and other related results are given in Section 7.1.

THEOREM 7.4. *Let M be a compact, orientable 3–manifold other than $S^1 \times S^2$ or a lens space. Assume that any two curves of ∂M which compress in M are on the same component of ∂M. Suppose that W is a genus two* *handlebody embedded in M such that W intersects every essential sphere in M at least three times and every essential disc at least two times. Suppose also that $N = M - \mathring{W}$ is irreducible. Let α and β be essential discs in W which cannot be isotoped to be disjoint. Assume that $M[\alpha]$ and $M[\beta]$ contain no essential disc which is contained in N and that $\partial\alpha$ and $\partial\beta$ do not compress in N.*

Then one of $M[\alpha]$ and $M[\beta]$ is irreducible and if both are irreducible then one is not a solid torus. Furthermore if $c_a \subset \partial M$ is a curve which compresses in $M[\alpha]$ and $c_b \subset \partial M$ is a curve which compresses in $M[\beta]$ then c_a and c_b cannot be isotoped to be disjoint.

1.4. Rational Tangle Replacement

Returning to a knot-theoretic interpretation of refilling meridians of a genus 2 handlebody, we can use sutured manifold theory to learn a great deal about knots or links which differ from an unknot or split link by a rational tangle. For the definitions of various types of tangles, I follow Eudave-Muñoz [**EM2**].

A **tangle** (B, τ) is a properly embedded pair of arcs τ in a 3–ball B. Two tangles (B, τ) and (B, τ') are **equivalent** if they are homeomorphic as pairs. They are **equal** if there is a homeomorphism of pairs which is the identity on ∂B. The **trivial tangle** is the pair $(D^2 \times I, \{-.5, .5\} \times I)$. A **rational tangle** is a tangle equivalent to the trivial tangle. Each rational tangle (B, r) has a disc $D_r \subset B$ separating the strands of r (each of which is isotopic into ∂B). The disc D_r is called a **trivializing** disc for (B, r). The **distance** $d(r, s)$

between two rational tangles (B,r) and (B,s) is simply the minimal intersection number $|D_r \cap D_s|$. We will often write $d(D_r,D_s)$ instead of $d(r,s)$. A **prime** tangle (B,τ) is one without local knots (i.e. every meridional annulus is boundary-parallel) and where no disc in B separates the strands of τ.

Given a knot $L_\beta \subset M$ and a 3–ball B' intersecting L_β in two arcs such that $(B', B' \cap L_\beta) = (B', r_\beta)$ is a rational tangle, to replace (B', r_β) with a rational tangle (B', r_α) is to do a **rational tangle replacement** on L_β. Notice that that $\eta(L_\beta) \cup B$ is a genus 2 handlebody W. The knots or links L_β and L_α can be obtained by refilling the meridians β and α respectively. If $M = S^3$ then $(B,\tau) = (S^3 - \mathring{B}', L_\beta - \mathring{B}')$ is a tangle. Figure 1.5 depicts a rational tangle replacement converting the unlink to the Hopf link and how to achieve this by boring. Notice that this rational tangle operation is simply a crossing change. Since $2d(\alpha,\beta) = \Delta(\partial\alpha,\partial\beta)$, for a crossing change $d = 2$. We will use the notation of this paragraph whenever we consider rational tangle replacement.

In [**EM2**], Eudave-Muñoz states the following related theorems. He proves theorems (EM 1) - (EM 3). Theorems (BS 4), (S 5), and (EM 6) were proven previously by Bleiler and Scharlemann [**BS1, BS2**], Scharlemann [**S1**], and Eudave-Munoz [**EM1**], respectively. Gordon and Luecke [**GLu2**] have given different proofs of Theorems (EM 1) - (EM 3).

THEOREM (Eudave-Muñoz). *Suppose that a rational tangle replacement of distance d on a knot or link L_β produces a knot or link L_α. Let (B,τ), (B', r_α) and (B', r_β) be as above.*

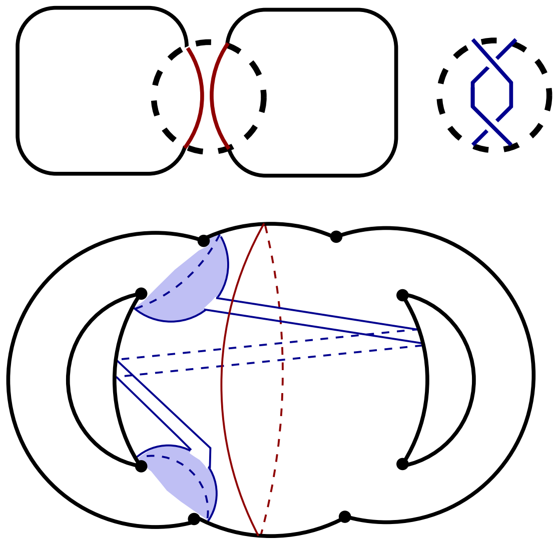

FIGURE 1.5. A rational tangle replacement converting the unlink to the Hopf link.

(EM 1) *If* (B, τ) *is prime and* L_α *and* L_β *are composite then* $d \leq 1$.

(EM 2) *If* (B, τ) *is prime, if* L_α *is a split link and if* L_β *is composite then* $d \leq 1$.

(EM 3) *If* (B, τ) *is any tangle and if* L_α *and* L_β *are split links, then* $r_\alpha = r_\beta$.

(BS 4) *If* (B, τ) *is a prime tangle and if* L_α *and* L_β *are both unknots, then* $r_\alpha = r_\beta$.

(S 5) *If* (B, τ) *is any tangle, if* L_β *is a trivial knot and if* L_α *is a split link then* (B, τ) *is a rational tangle and* $d \leq 1$.

(EM 6) *If* (B, τ) *is prime, if* L_β *is a composite knot or link and if* L_α *is the unknot, then* $d \leq 1$.

The work in this paper can be used to give new proofs of all but the first. In fact, we give two new proofs of Theorems (EM 2), (EM 3) and (S 5).

The histories of (S 5) and (EM 6) are interesting. Consider a split link L in S^3 with components L_0 and L_1 and an embedding $b\colon I \times I \to S^3$ so that $b(I \times \{i\})$ is contained on L_i for $i \in \partial I$ and so that $b(I \times \mathring{I})$ is disjoint from L. We can form a knot $K = L_0 \#_b L_1$ by forming the **band sum** of L_0 and L_1 using the band b. K is defined to be

$$K = \big(L - b(I \times \partial I)\big) \cup b(\partial I \times I).$$

See Figure 1.6 for an example.

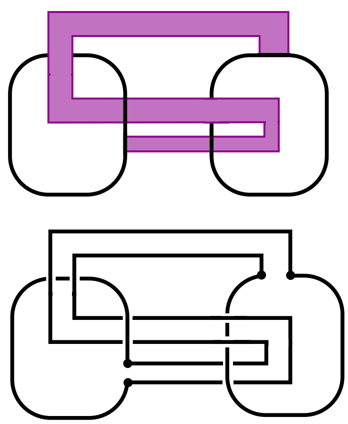

FIGURE 1.6. A band sum creating the granny knot

We often ignore the distinction between the function b and its image. If the band intersects a splitting sphere for L in a single arc then K is the connected sum of L_0 and L_1 and b is a trivial band. By looking at a regular neighborhood of $b(I \times [.25, .75])$, K and L are easily seen to differ by rational tangles distance 1 apart.

Matumoto [**K2**, Problem 1.2 A] asked: if K is the unknot, must L be the unlink and b a trivial band? Scharlemann [**S1**] answered this in the affirmative using a purely combinatorial argument. Later, Gabai and Scharlemann independently and simultaneously proved that the genus of K is at least the sum of the genera of L_0 and L_1, answering a question of Lickorish [**K2**, Problem 1.1]. Gabai's proof [**G4**] was a simple application of his sutured manifold theory [**G1, G2, G3**] and a trick of Abby Thompson. Scharlemann's proof [**S3**] was an application of combinatorial sutured manifold theory, his de-foliated version of Gabai's machinery. It is fairly easy to see that the statement of (S 5) includes Scharlemann's original band sum theorem (see Section 8.4). The methods of this paper also give a new proof of Gabai and Scharlemann's theorem on the superadditivity of genus under band sum.

The unknotting number of a knot is the minimal number of crossing changes necessary to convert the knot into the unknot. It has long been conjectured that unknotting number is additive with respect to connected sum [**K2**, Problem 1.69 B]. A weaker conjecture (due to de Souza) is that the connected sum of n knots has unknotting number at least n [**K2**, Problem 1.69 A]. For $n = 2$, this was proven by Scharlemann [**S2**] using a completely combinatorial argument. It was later reproven by Scharlemann and Thompson [**ST1**] using combinatorial sutured manifold theory. Theorem (EM 6) is a generalization of this fact that was proven completely combinatorially (without sutured manifold theory). The present work continues the tradition of using sutured manifold theory to reprove and extend theorems originally proved combinatorially. The methods of this paper have the added advantage that, in some circumstances, they significantly simplify previously existing sutured manifold theory proofs, for example the proof that an unknotting number one knot is prime.

Generalized crossing changes (see Figure 1.7) are another type of rational tangle replacment. These have been extensively studied by Scharlemann and Thompson [**ST1**] and Lackenby [**L1, L3**] using a Dehn surgery description of generalized crossing changes. Since the inequalities I obtain are similar to Lackenby's, I will briefly summarize one of his results.

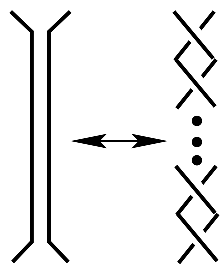

FIGURE 1.7. A generalized crossing change

A crossing disc $D \subset S^3$ for a knot $K \subset S^3$ is a disc which is intersected by the knot exactly twice with intersection number zero. Let $L = \partial D$ be the **crossing link**. Performing $\pm 1/n$ Dehn-surgery ($n \in \mathbb{N}$) on L (using meridian/longitude coordinates) produces a generalized crossing change of order n using L. Notice that a generalized crossing change of order n can also be described as a rational tangle replacement of one tangle by a tangle of distance $d = 2n$ away. A **Seifert surface** for a knot or link L is an orientable surface S without closed components for which $\partial S = L$. We will usually work with the surface $S - \mathring{\eta}(L)$ which we also refer to as a Seifert surface.

A consequence of Lackenby's result [**L1**, Corollary 3.5] is:

THEOREM. *Let K be a non-trivial knot in S^3 and K' a knot obtained by a generalized crossing change of order $d/2 > 1$ using L. Suppose that the genus of K' is strictly less than that of K and that F is a properly embedded* *orientable surface in the exterior of K. Then there is an ambient isotopy of L in $S^3 - \mathring{\eta}(K)$ so that after the isotopy*

$$-\chi(F) \geq (d-1)|F \cap L|$$

Lackenby's work actually applies to Dehn twists about knots other than those bounding crossing discs. The results of this paper provide similar, but more limited, information about knots and links obtained from a split link or unknot by boring or rational tangle replacement. Here are simplified versions of two theorems for boring operations more general then rational tangle replacement. Section 1.5 describes some results pertaining to Dehn Surgery.

The first theorem turns out to be related to two theorems of Scharlemann and Thompson. The first [**ST1**] states that either a satellite torus for a knot can be isotoped to be disjoint from a given crossing disc or there is a minimal genus Seifert surface for the new knot which intersects the crossing link in no more than two points. (Lackenby's previously mentioned result is closely related to this fact.) The second related theorem of Scharlemann and Thompson [**ST2**] states that a tunnel for a tunnel number one knot can be isotoped and slid to be disjoint from a minimal genus Seifert surface. These connections are explained more in Section 7.

THEOREM 7.5. *Suppose that L_α is a knot or link in S^3 obtained by boring a knot or link L_β using handlebody W. Suppose that either α is non-separating or $\partial W - \partial\alpha$ is incompressible in N. Suppose also that one of the following holds:*

- *L_β is an unknot*
- *L_β is a split link and $\partial W - \partial\beta$ is incompressible in N.*

Then there is a minimal genus Seifert surface for L_α which is disjoint from $\overline{\alpha}$.

The second theorem, at the cost of putting more hypotheses on the embedding of W in $M = S^3$, studies circumstances guaranteeing that a knot or link L_β obtained by a split link (for example) is hyperbolic.

THEOREM 7.8. *Suppose that L_β is a knot or link obtained by boring the link L_α using a handlebody $W \subset S^3$ with $N = S^3 - \mathring{W}$ boundary-irreducible. Suppose that L_α is a split link or that there is no minimal genus Seifert surface for L_α disjoint from $\overline{\alpha}$. If the exterior of L_β contains an essential annulus or torus then one of the following holds:*

(1) There is an essential torus in N

(2) There is an essential annulus in the exterior of L_β which is disjoint from $\overline{\beta}$ and which is either disjoint from or has meridional boundary on some component of L_β.

(3) $\Delta = 2$ and if there is an essential annulus then there is one which is either disjoint from or has meridional boundary on some component of L_β.

Returning to rational tangle replacement here are two theorems similar to Lackenby's. The first is a restatement of Theorem 7.7 for rational tangle replacement.

SIMPLIFIED COROLLARY 8.4. *Suppose that $L_\beta \subset S^3$ is obtained by a rational tangle replacement of distance $d \geq 1$ on a split link or unknot L_α. Assume that (B,τ) is prime and that there does not exist an essential disc in the exterior of L_α which is disjoint from $\overline{\alpha}$. Then L_β has a minimal genus Seifert surface $\overline{Q}$ disjoint from $\overline{\beta}$ such that one of the following holds:*

- *$\overline{\beta}$ is properly isotopic into $\overline{Q}$*
- *$-\chi(\overline{Q}) \geq d$ and L_α is a split link*
- *$-\chi(\overline{Q}) \geq d-1$ and L_α is an unknot.*

An example is given which shows that the first possibility cannot be eliminated.

We can also obtain an inequality similar to Lackenby's for studying essential planar surfaces with meridional boundary in the exterior of a knot L_β.

THEOREM 8.6. *Suppose that the knot or link L_β is obtained from a knot or link L_α by a rational tangle replacement of distance $d \geq 1$. Suppose that (B,τ) is prime and that L_α is a split link or does not contain a minimal genus Seifert surface disjoint from the arc $\overline{\alpha}$. If L_β has an essential properly embedded meridional planar surface with m boundary components, then it contains such a surface $\overline{Q}$ with $|\partial\overline{Q}| \leq m$ such that either $\overline{Q}$ is contained in B or*

$$|\overline{Q} \cap \overline{\beta}|(d-1) \leq |\partial\overline{Q}| - 2.$$

1.5. Dehn Surgery Results

The results of this paper can be applied to surfaces other than essential spheres, discs, meridional planar surfaces, and Seifert surfaces. Planar surfaces and punctured tori with non-meridional boundary are a particularly interesting class of surfaces. Theorems about such surfaces can often be translated into statements about the results of Dehn surgery on such a knot or link.

The cabling conjecture postulates that surgery on a non-trivial knot in S^3 produces a reducible manifold only if the knot is cabled and the surgery slope is the slope of the cabling annulus. Gordon and Luecke [**GLu1**] have shown that if a knot has a reducing surgery then the surgery slope is an integer. The answer to the question of what 2–component links have reducing surgeries is likely much more complicated. Reducing surgeries on 2–component links are easy to create: most every surgery on a split link in S^3 produces a reducible manifold. If the surgery slopes are integers and a Kirby band-move is performed, the resulting link is likely not a split link but still has a surgery producing a reducible manifold. Another way of creating such a 2–component link is to take a knot with a reducing surgery as one component and take any knot in its complement with the meridional surgery as the other component. One could then perform a Kirby band-move on these knots, producing a still more complicated 2–component link with a reducing surgery.

More complicated than the cabling conjecture is the question of what Dehn surgeries on what hyperbolic knots in S^3 will produce a manifold containing an essential torus. Gordon and Luecke [**GLu3**] have shown that such a Dehn surgery slope must be either an integer or half an integer. Furthermore, they have shown [**GLu4**] that the only hyperbolic knots with half integer surgery slope producing a toroidal manifold are the knots and surgeries described by Eudave-Muñoz [**EM5**].

If Dehn surgery on a hyperbolic knot or link K with slope r (if K is a link with n components, r is an n-tuple of slopes, one on each component) produces a reducible or toroidal manifold it is not difficult to show that there is, in the complement of K, an essential planar surface or punctured torus whose slope on K is the the surgery slope. The final result we shall mention here in the introduction concerns the possibilities for essential planar surfaces and punctured tori in the exterior of a knot or link obtained by rational tangle replacement on a split link. The theorem is not sufficient for understanding reducing and toroidal surgeries on such a knot or link due to the possibility of the second conclusion. It may, however, be helpful for understanding Dehn surgery on a strongly invertible knot or link. Hirasawa and Shimokawa [**HS**], for example, proved that if attaching a band to a non-trivial $(2,2p)$ torus link produces an unknot then the band is "standard", i.e. can be isotoped into the essential annulus. This is used to prove that no Dehn surgery on a strongly invertible knot can yield the lens space $L(2p,1)$ for any $p \in \mathbb{Z}$.

SIMPLIFIED THEOREM 8.8. *Suppose that L_β is a knot or link obtained by rational tangle replacement of distance d on a knot or link L_α. Suppose that (B,τ) is prime and that L_α is a split link or does not contain a minimal* *genus Seifert surface disjoint from $\overline{\alpha}$. Then if L_β contains an essential planar surface or punctured torus in its exterior there is such a surface $\overline{Q}$ such that one of the following holds:*

- *L_β is a link and $\partial\overline{Q}$ is disjoint from some component of L_β.*
- *$\overline{Q}$ is disjoint from $\overline{\beta}$ and $\overline{\beta}$ is isotopic into $\overline{Q}$.*
- *$\overline{Q}$ has meridional boundary on some component of L_β.*
- *$d \leq 3$.*

In the non-simplified version of the theorem, much more information is given concerning the last case.

1.6. Technical Advances

Other interesting aspects of the present work are certain technical advances pertaining to sutured manifold theory and combinatorial methods in the study of 2–handle addition. An overview of combinatorial sutured manifold theory is given in Section 2; for the moment some familiarity with the theory is assumed.

Vaguely speaking, the significance of the sutured manifold theory results in this thesis is that they "relativize" previously existing methods in sutured manifold theory. Combinatorial sutured manifold theory has often relied on certain (non-empty) 1–complexes properly embedded in the manifold. For all previous applications (that I am aware of) the 1–complex has been either a knot [**S3, S4**], an edge with a loop at each vertex [**S3, EM3, EM4**], or a single vertex with two loops attached [**ST1, Ko, EM4**]. Alternatively, many other sutured manifold theory results [**G4, S3, L2**] have not used a

1–complex at all, but have instead taken a sutured manifold hierarchy to be disjoint from a certain torus boundary component. At the end of the hierarchy, a solid torus is attached to that torus component and the results are analyzed. Both philosophies are present in the current work. The "first sutured manifold theorem" does not (in principle) use a 1–complex and studies hierarchies which are disjoint from a certain annulus in the boundary of the manifold. (For technical reasons, however, the proof is best written using a 1–complex.) At the end of the hierarchy the result of attaching a 2–handle to the annulus is analyzed. The "second sutured manifold theorem" takes the 1–complex in the sutured manifold to be an arc. Theorem 9.1 of [**S3**] is adapted by replacing the knot in that theorem with the arc. As part of that process, certain well-known combinatorial structures (e.g. Scharlemann cycles) are adapted and reworked.

The first sutured manifold result is usually more powerful, but the second sutured manifold result does have its uses. Some of these uses are explored in Section 9.

Before the first and second sutured manifold results are stated and proved, a quick overview of combinatorial sutured manifold theory is given.

CHAPTER 2

Combinatorial Sutured Manifold Theory

2.1. Introducing Sutured Manifold Theory

In [**S3**], Scharlemann introduced a combinatorial version of Gabai's sutured manifold theory [**G1, G2, G3**]. A much fuller exposition of combinatorial sutured manifold theory can be found in Scharlemann's paper. In this introduction, I focus only on those aspects which will be used in what follows. The notation is chosen to correspond to that used by Scharlemann. It is not necessarily the notation which will be used later. For example, in this section β will be a 1–complex, but in later sections β will be a disc in a genus 2 handlebody.

DEFINITION. A **sutured manifold** (M,γ,β) consists of a compact oriented 3–manifold M, a collection of oriented simple closed curves $\gamma \subset \partial M$, and a finite 1–complex $\beta \subset M$. Either γ or β may be the empty set. Let $A(\gamma) = \eta(\gamma)$ and let $T(\gamma)$ be a collection of tori in ∂M which are disjoint from γ. If ∂M is non-empty we require $\text{cl}(\partial M - (\gamma \cup T(\gamma))$ to consist of two (possibly disconnected) surfaces each with boundary equal to γ and whose intersection is exactly γ. The intersection of one of these surfaces with $\partial M - A(\gamma)$ is called $R_+ = R_+(\gamma)$ and the intersection of the other surface with $\partial M - A(\gamma)$ is called $R_- = R_-(\gamma)$. (See Figure 2.4.) We consider the surfaces to be given normal orientations so that R_+ has outward pointing normal and R_- has inward pointing normal.

We require the 1–complex β to be properly embedded in M; that is, $\partial\beta = \beta \cap \partial M$ consists of the valence 1 vertices of β. We say that M has the sutured manifold structure (M, γ, β), often abbreviated to (M, γ) when β is unambiguous. The notation $R_\pm$ will indicate $R_+(\gamma)$ or $R_-(\gamma)$ and $R(\gamma)$ will indicate $R_+ \cup R_-$. In this paper, β will either be empty or will be a properly embedded arc.

Sutured manifold theory is most useful when $H_2(M, \partial M)$ is non-trivial. Note that this is always the case when $\partial M \neq \varnothing$. If ∂M consists entirely of tori then M has a sutured manifold structure with $\gamma = \varnothing$.

3–manifolds containing incompressible surfaces have long been studied by using hierarchies. Sutured manifold theory studies hierarchies of sutured 3–manifolds. The theory is both more powerful, and more complicated than, typical hierarchy arguments. The main tool for studying and using hierarchies of sutured manifolds is the Thurston norm, or more generally, a β-norm.

DEFINITION. For a compact connected surface $S \subset M$ in general position with respect to the 1–complex β, let

$$\chi_\beta(S) = \max\left(0, |S \cap \beta| - \chi(S)\right)$$

where $\chi(S)$ denotes the euler characteristic of S. For a disconnected compact surface S let $\chi_\beta(S)$ be the sum of $\chi_\beta(S_i)$ over all components S_i of S.

For a class $a \in H_2(M,X)$ define

$$\chi_\beta(a) = \inf\{\chi_\beta(S) : S \text{ is an embedded representative of } a\}.$$

If $\beta = \varnothing$, then $\chi_\beta : \ H_2(M,X) \to \mathbb{Z}_+$ is the **Thurston norm**, otherwise it is called a β-norm.

Of utmost importance is the notion of β–tautness for both surfaces in a sutured manifold (M,γ,β) and for a sutured manifold itself.

DEFINITION. Let S be a properly embedded surface in M.

- S is β–minimizing in $H_2(M,\partial S)$ if $\chi_\beta(S) = \chi_\beta[S,\partial S]$.
- S is β–incompressible if $S-\beta$ is incompressible in $M-\beta$.
- S is β–taut if it is β–incompressible, β–minimizing in $H_2(M,\eta(\partial S))$ and each edge of β intersects S with the same sign. If $\beta = \varnothing$ then we say either that S is $\varnothing$–taut or that S is taut in the Thurston norm.

DEFINITION. (M,γ,β) is β–taut if

- $\partial\beta$ is disjoint from $A(\gamma) \cup T(\gamma)$
- $T(\gamma)$, $R_+(\gamma)$, and $R_-(\gamma)$ are all β–taut.
- M is β–irreducible; that is, $M-\beta$ is irreducible.

Notice that if (M,γ) is β–taut then $\chi_\beta(R_+) = \chi_\beta(R_-)$ and no edge of β has both endpoints in $R_\pm$. If $\beta = \varnothing$ we will often abbreviate "$\varnothing$–taut" to simply "taut". Figure 2.1 depicts several easy examples. In A) the manifold is a 3–ball with a single suture on its boundary. It is $\varnothing$–taut. In B) (M,γ)

is a 3–ball with a single suture on ∂M and arcs β joining R_+ to R_-. It is both β–taut and $\varnothing$–taut. Example C) is similar to B) except that there are three sutures on ∂M. In this case (M,γ) is β–taut but not $\varnothing$–taut. Example D) depicts a solid torus with two parallel sutures on the boundary. As long as the sutures are not meridians of the solid torus, the sutured manifold is $\varnothing$–taut.

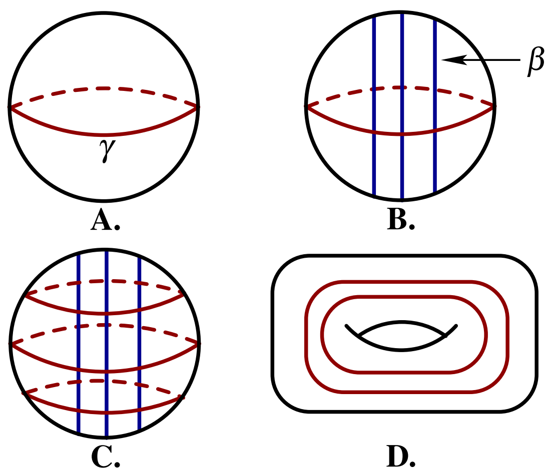

FIGURE 2.1. Four easy examples

This thesis is most interested in the situation when β is an arc properly embedded in a 3–manifold. The following are, therefore, important examples of sutured manifolds. Their claims follow easily from the preceding definitions.

EXAMPLE. Let M be a compact, oriented 3–manifold with toral boundary and let T_1 and T_2 be distinct torus components of ∂M. Let β be a properly embedded arc in M with an endpoint on each of T_1 and T_2 and let b be a meridian curve on $\partial\eta(\beta)$. Suppose that $M - \mathring{\eta}(\beta)$ is irreducible and that $\partial(M - \mathring{\eta}(\beta)) - b$ is incompressible in $M - \mathring{\eta}(\beta)$. Let $\gamma = \varnothing$, $T(\gamma) =$

$\partial M - (T_1 \cup T_2)$. Let $R_+ = T_1$ and $R_- = T_2$. Then (M, γ, β) is β–taut (Figure 2.2.A) and $(M - \mathring{\eta}(\beta), b)$ is $\varnothing$–taut (Figure 2.2.B).

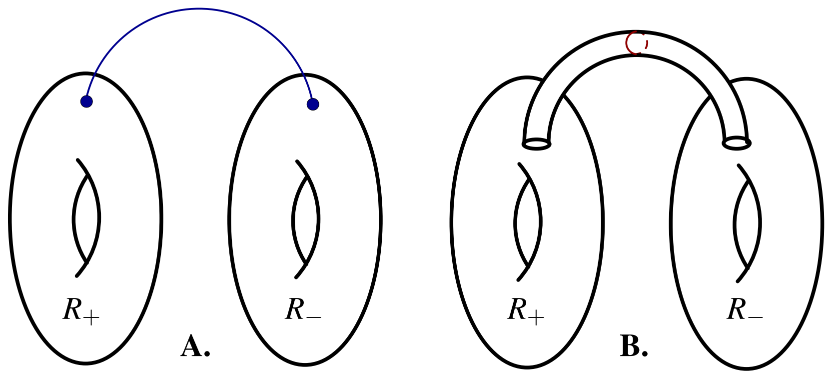

FIGURE 2.2. When β is an arc joining two distinct boundary components

EXAMPLE. Let M be a compact, oriented 3–manifold with toral boundary. Let T_1 be a torus component of ∂M. Let β be a properly embedded arc in M with endpoints on T_1 and let b be a meridian curve on $\partial \eta(\beta)$. Suppose that $M - \mathring{\eta}(\beta)$ is irreducible. Choose parallel curves $\gamma \subset T_1$ which separate the endpoints of β. If $\partial(M - \mathring{\eta}(\beta)) - (b \cup \gamma)$ is incompressible in $M - \mathring{\eta}(\beta)$ then (M, γ, β) is a β–taut sutured manifold (Figure 2.3.A). Also, $(M, \gamma \cup b)$ is taut (Figure 2.3.B).

Since we are interested in hierarchies of sutured manifolds we need to specify the sorts of surfaces along which we will be decomposing our sutured manifolds.

DEFINITION. Suppose that (M, γ, β) is a sutured manifold.

(1) A **conditioned surface** $S \subset M$ is an oriented properly embedded surface such that:

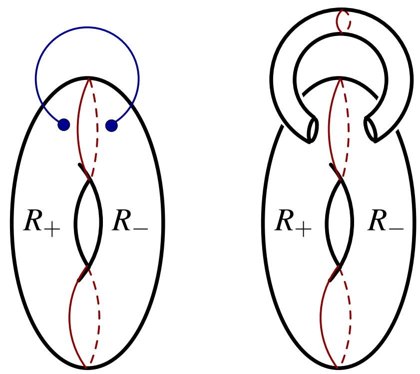

FIGURE 2.3. When β is an arc joining a boundary component to itself

- If T is a component of $T(\gamma)$ then $\partial S \cap T$ consists of coherently oriented parallel circles.
- If A is a component of $A(\gamma)$ then $S \cap A$ consists of either circles parallel to γ and oriented the same direction as γ or arcs all oriented in the same direction.
- No collection of simple closed curves of $\partial S \cap R(\gamma)$ is trivial in $H_1(R(\gamma), \partial R(\gamma))$.
- Each edge of β which intersects $S \cup R(\gamma)$ does so always with the same sign.

(2) A **product disc** is a disc $I \times I \subset M - \mathring{\eta}(\beta)$ such that $I \times \{0\} \subset R_+(\gamma)$, $I \times \{1\} \subset R_-(\gamma)$, and $\{0,1\} \times I \subset A(\gamma)$. See Figure 2.4.

(3) A **product annulus** is an annulus $S^1 \times I \subset M - \mathring{\eta}(\beta)$ such that $S^1 \times \{0\} \subset R_+$, and $S^1 \times \{1\} \subset R_-$. A product annulus is **β-nontrivial** if it cannot be extended to an embedding $D^2 \times I \subset M - \mathring{\eta}(\beta)$ with $D^2 \times \{0\} \subset R_+$ and $D^2 \times \{1\} \subset R_-$. See Figure 2.5.

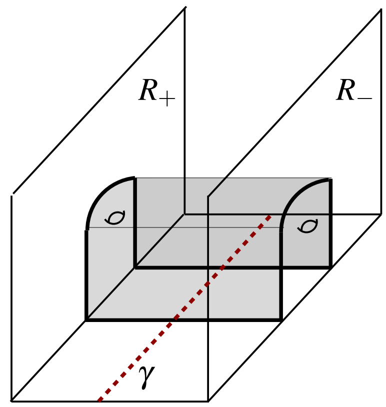

FIGURE 2.4. A product disc.

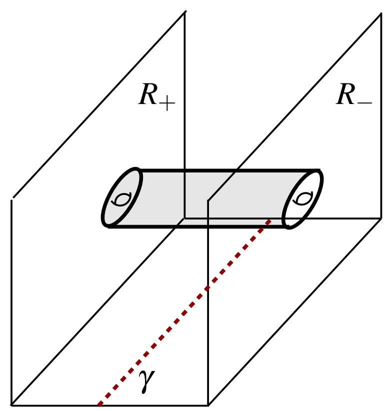

FIGURE 2.5. A product annulus.

If (M, γ, β) is a sutured manifold and $S \subset M$ is a conditioned surface, product disc, or β-nontrivial product annulus, the manifold $M' = M - \mathring{\eta}(S)$ inherits a natural sutured manifold structure (M', γ', β'). The 1–complex β' is simply $\beta - \mathring{\eta}(S)$; we will often continue to refer to β' as β. The sutures γ' are obtained by taking the "oriented double-curve sum" of ∂S and γ. See Figure 2.6 for an example and refer to **[G1, S3]** for more details. (The assumption that S is a conditioned surface, product disc, or product annulus is not strictly necessary, the weaker assumption that S is a "decomposing surface" [**S3**, Definition 2.3] will do, but this notion is not necessary for this paper.) We say that $(M,\gamma) \xrightarrow{S} (M',\gamma')$ is a **sutured manifold decomposition**. It is a β–taut sutured manifold decomposition if (M,γ) is β–taut and (M',γ') is β'–taut.

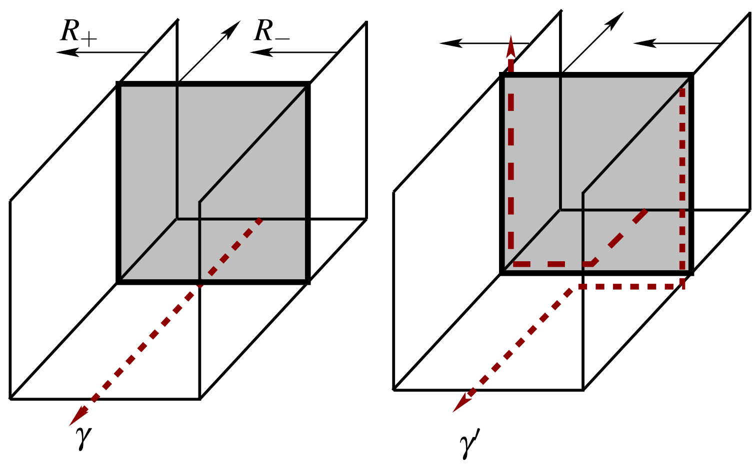

FIGURE 2.6. Forming γ' by decomposing along a surface

Conditioned surfaces and decompositions along them play an important role in this paper, so it will be useful to note the following theorem and some aspects of its proof.

THEOREM 2.1. *Let (M,γ) be a β–taut sutured manifold and let y be a non-trivial element of $H_2(M,\partial M)$. Then there exists a conditioned surface $(S,\partial S) \subset (M,\partial M)$ containing no closed components such that $[S,\partial S] = y$. Furthermore, S is β–taut and the decomposition of M along S is β–taut.*

PROOF. This is a combination of Theorems 2.5 and 2.6 of [**S3**]. The surface S is formed by beginning with a surface σ in M, representing y, such that $\partial\sigma$ fulfills the requirements for the boundary of a conditioned surface. (That such a surface exists is a consequence of [**S3**, Theorem 2.5].)

The required surface S is then formed by taking the oriented sum of S with some number of copies of R_+ and some number of copies of R_-. $\square$

DEFINITION. A β–taut sutured manifold hierarchy is a finite sequence of β–taut sutured manifold decompositions

$$(M_0,\gamma_0) \xrightarrow{S_1} (M_1,\gamma_1) \xrightarrow{S_2} \dots \xrightarrow{S_n} (M_n,\gamma_n)$$

for which

(1) each S_i is a conditioned surface, product disc, or β-nontrivial product annulus
(2) If either end of a product annulus S_{i+1} bounds a disk in $R(\gamma_i)$ then no component of β which intersects the disk is an edge isotopic into the annulus
(3) $H_2(M_n,\partial M_n)=0$, implying that ∂M_n is a union of spheres.

We now state the two fundamental theorems of combinatorial sutured manifold theory. The first states that β–tautness can be carried down a hierarchy and the second (perhaps, the more amazing) states that β–tautness can be carried up a hierarchy.

THEOREM 2.2 (Theorem 4.19 of [**S3**]). *Every β–taut sutured manifold admits a β–taut sutured manifold hierarchy. For a given $a \in H_2(M,\partial M)$, the hierarchy can be chosen so that the first surface in the hierarchy represents a.*

THEOREM 2.3 (Corollary 3.9 of [**S3**]). *Suppose that*

$$(M_0,\gamma_0) \xrightarrow{S_1} (M_1,\gamma_1) \xrightarrow{S_2} \dots \xrightarrow{S_n} (M_n,\gamma_n)$$

is a sequence of sutured manifold decompositions in which

- *no component of M_0 is a solid torus disjoint from β and γ_0*
- *each S_i is either a conditioned surface, a product disc, or a β-nontrivial product annulus.*
- *no closed component of any S_i separates.*

Then if (M_n, γ_n) is β–taut, every decomposition in the series is β–taut.

Typically these two theorems are used in conjunction. To illustrate this here are rough outlines (including several serious imprecisions) of the main sutured manifold theorems of this paper. For the first (Section 3.1), suppose that $(N, \gamma \cup a)$ is a $\varnothing$–taut sutured manifold and that $a \subset \partial N$ is an essential simple closed curve. Take a $\varnothing$–taut sutured manifold hierarchy of N

$$(N, \gamma \cup a) = (N_0, \gamma_0 \cup a) \xrightarrow{S_1} (N_1, \gamma_1 \cup a) \xrightarrow{S_2} \dots \xrightarrow{S_n} (N_n, \gamma_n \cup a)$$

except instead of stopping when $H_2(N_n, \partial N_n) = 0$, stop when $H_2(N_n, \partial N_n - \mathring{\eta}(a)) = 0$. That is, cut along conditioned surfaces, product discs, and product annuli disjoint from a as much as possible, and then stop. It turns out that such a modified notion of hierarchy exists. Attach a 2–handle to a in ∂N_n and examine what happens. In an ideal world, Theorem 2.3 would tell us that if $(N_n[a], \gamma_n)$ is $\varnothing$–taut then so is $(N[a], \gamma)$ unless a component of $N[a]$ is a solid torus disjoint from γ. A moment's thought however shows that, as phrased, the hypotheses that the surfaces S_i be conditioned surfaces, product discs, or product annuli in $(N_{i-1}[a], \gamma_i)$ for Theorem 2.3 may not be satisfied. In the proof of the first sutured manifold theorem, a more subtle argument is used. The argument still relies on Theorem 2.3.

For the second sutured manifold theorem (Section 3.2), let (N, γ, β) be a β–taut sutured manifold with β a properly embedded arc. Take a β–taut sutured manifold hierarchy of N

$$(N,\gamma) \xrightarrow{S_1} (N_1,\gamma_1) \xrightarrow{S_2} \ldots \xrightarrow{S_n} (N_n,\gamma_n)$$

stopping when $H_2(N_n, \partial N_n) = 0$. The sutured manifold (N_n, γ_n, β_n) is β_n–taut. A combinatorial argument at this final stage will show that (in certain circumstances) (N_n, γ) is also $\varnothing$–taut. Then Theorem 2.3 shows that, unless a component of N is a solid torus disjoint from γ and β, (N, γ) is $\varnothing$–taut.

The main tools needed for making combinatorial arguments are parameterizing surfaces.

DEFINITION. A parameterizing surface Q in a sutured manifold (M, γ, β) is a surface $(Q, \partial Q) \subset (M - \mathring{\eta}(\beta), \partial(M - \mathring{\eta}(\beta)))$ such that no component of Q is a disc with boundary in $R_\pm$.

We would like to be able to manage the the interactions between a parameterizing surface and a sutured manifold hierarchy. Fortunately, this can be done, perhaps at the cost of slightly changing the hierarchy. The details are slightly complicated and not terribly relevant for what follows, so we summarize the main points. Suppose that $(M, \gamma, \beta) \xrightarrow{S} (M', \gamma', \beta')$ is a β–taut sutured manifold decomposition. If S is a conditioned surface, then the decomposition **respects** Q if $Q \cap M'$ is still a parameterizing surface. This can always be arranged by replacing S with the surface obtained by taking the double curve sum of S and some number of copies of R_+ and some number of copies of R_- [**S3**, Lemma 7.5]. If S is a product disk or β-nontrivial product annulus we also want to arrange the decomposition so that it "respects" Q. This can be done after isotoping S and Q, boundary-compressing Q using discs contained in S, and then removing discs of Q with boundary contained in $R(\gamma)$. After such operations the surface $Q' = Q - \mathring{\eta}(S)$ is then a parameterizing surface for M'. We say that a β–taut hierarchy respects Q if at each stage Q' is formed by the processes just described. By Theorem 7.8 of [**S3**] we may assume that a β–taut sutured manifold hierarchy respects a given parameterizing surface. Even though the parameterizing surface Q_n at the end of a hierarchy may not be a subset of Q (due to product discs and annuli), Lackenby [**L1**] notes that there is a collection of discs $\mathscr{D} \subset Q_n$ such that each disc in $\mathscr{D}$ is a regular neighborhood of a point in ∂Q_n and $Q_n - \mathscr{D} \subset Q$. Thus it is easy to take information about Q_n and translate it into information about Q.

To a parameterizing surface Q we associate a number $I(Q)$ called the index of Q. Let ψ be a compact 1-manifold (possibly with boundary) embedded (but not necessarily properly embedded) in $\partial(M - \mathring{\eta}(\beta))$. Assume that ψ is in general position with respect to γ. Define $\mu(\psi)$ to be the number of essential arcs of $\psi \cap \eta(\mathscr{E})$ where $\mathscr{E}$ is the set of edges of β. (Notice that if β is a single loop, then $\mu(\psi) = 0$.) Define $\nu(\psi)$ to be the number of essential arcs of $\psi \cap A(\gamma)$. The **index** of Q is then defined to be $I(Q) = \mu(\partial Q) + \nu(\partial Q) - 2\chi(Q)$. In [**S3**], $I(Q)$ has an additional term $\mathscr{K}$. This is a function, which can be chosen somewhat arbitrarily, on arcs passing through vertices of β. Since in this work, we choose $\mathscr{K}$ to be zero, we make no further mention of it.

EXAMPLE. Figure 2.7 depicts a portion of a sutured manifold with a parameterizing surface. In the figure, four pieces of sutures are shown and β consists of four arcs. The surface Q is a twice-punctured torus. Each boundary component of Q crosses the sutures four times and crosses arc components of β twice. Thus $I(Q) = 4 + 8 - 2(-2) = 16$.

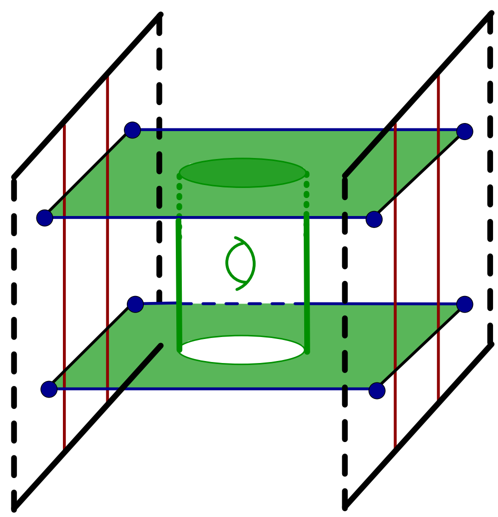

FIGURE 2.7. The parameterizing surface is a twice-punctured torus.

The usefulness of the index comes from the following theorem.

THEOREM 2.4 ([**S3**, Lemmas 7.5 and 7.6]). *Suppose that*

$$(M, \gamma, \beta) \xrightarrow{S} (M', \gamma', \beta')$$

is a β–taut sutured manifold decomposition adapted to the parameterizing surface $Q \subset M - \mathring{\eta}(\beta)$ with Q' the resulting parameterizing surface in M'. Then $I(Q') \leq I(Q)$. In fact, if S is a conditioned surface then $I(Q') = I(Q)$.

There are two very simple types of parameterizing surfaces which are of particular importance. Suppose that b is an arc component of β and that Q is a disc with boundary consisting of two arcs, one an essential arc in $\partial\eta(\beta)$ and the other an arc on ∂M which crosses exactly one suture. Then Q is said to be a **cancelling disc** for b. See Figure 2.8 for an example. Suppose that b' is also an arc component of β (possibly equal to b). If ∂Q consists of four arcs, one an essential arc in $\partial\eta(b)$, one an essential arc in $\partial\eta(b')$ and two arcs in $R(\gamma)$ then Q is said to be an **amalgamating disc** for b. If $b' \neq b$ it is a **(non-self) amalgamating disc** for b. Figure 2.9 depicts both a (non-self) amalgamating disc and a self-amalgamating disc. Notice that if Q is a connected parameterizing surface with $I(Q) = 0$ then either Q is an annulus or torus disjoint from $\gamma \cup \eta(\beta)$ or it is a disc disjoint from $\eta(\beta)$ or it is a cancelling or amalgamating disc for some arc of β.

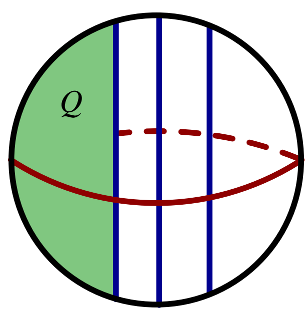

FIGURE 2.8. Q is a cancelling disc.

2.2. Conversing with Sutured Manifold Theory

Although sutured manifold theory is interesting in its own right, we would like to be able to translate conclusions about sutured manifolds and β–taut

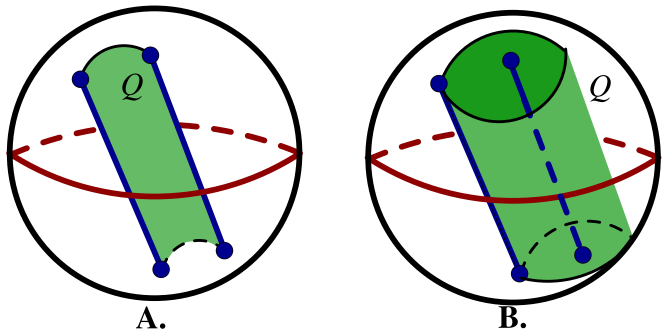

FIGURE 2.9. A) Q is a (non-self) amalgamating disc. B) Q is a self-amalgamating disc.

conditioned surfaces into conclusions which do not need to use the language of sutured manifold theory.

We will need two different methods for converting from a β-norm to the Thurston norm. Here are two methods for doing so. The first converts an arc component of β into a suture. When β consists of a single arc, we can use tautness in the Thurston norm of $M - \eta(\beta)$ (with an additional suture) to conclude that M is taut in the β-norm.

LEMMA 2.5 ([**S4**, Lemma 2.3]). *Suppose that (M, γ, β) is a sutured manifold with b an arc component of β having one end in each of $R_\pm$. Let $M' = M - \mathring{\eta}(b)$. and let γ' be γ together with a meridional curve on the boundary of the regular neighborhood of b. Then (M, γ, β) is β–taut if and only if $(M', \gamma', \beta - b)$ is $(\beta - b)$–taut.*

The other method of converting from a β-norm to the Thurston norm is most useful at the end of a β–taut hierarchy. We often hope to achieve $\varnothing$–tautness by showing that β is a collection of arcs in the final stage of the hierarchy and each arc can be cancelled using a cancelling disc or (non-self) amalgamating disc.

LEMMA 2.6 ([**S3**, Lemmas 4.3 and 4.4]). *Suppose that* (M,γ,β) *is a* β*–taut sutured manifold and that b is an arc component of* β *lying on a cancelling disc or (non-self) amalgamating disc. Then* $(M,\gamma,\beta-b)$ *is a* $(\beta-b)$*–taut sutured manifold.*

Torus components of ∂M which are disjoint from β may or may not have sutures, as desired. Since, however, a higher genus component of ∂M may not be β-minimizing, it may be necessary to place sutures on those components in order to give M a β–taut sutured manifold structure. Techniques for doing so are described in [**S4**] and [**L2**]. We will ultimately need a slight variation of those results, but that discussion is deferred until we have described the specific sutured manifolds of interest in this paper.

CHAPTER 3

Adding a 2–handle to a sutured manifold

This section describes two methods for proving that the manifold $N[a]$ obtained by adding a 2–handle to a curve a on a genus two or greater boundary component F of a compact, connected, orientable 3–manifold N is taut. We denote the core of the 2–handle by α so that $\partial\alpha = a$. The cocore of the 2–handle $\eta(\alpha)$ is an arc $\overline{\alpha}$. Suppose that sutures $\gamma \subset \partial N$ disjoint from a have been chosen so that $(N, \gamma \cup a)$ is a taut sutured manifold, or, equivalently (Lemma 2.5), so that $(N[a], \gamma)$ is an $\overline{\alpha}$–taut sutured manifold.

Suppose that $\mathscr{B} = \{b_1, \ldots, b_{|\mathscr{B}|}\}$ are pairwise disjoint, pairwise nonparallel essential curves in F each of which intersects $a \cup \gamma$ minimally. Suppose that $Q \subset N$ is a surface with q_i boundary components parallel to b_i. Let $\partial_0 Q$ denote the boundary components of Q which are not parallel to any curve in $\mathscr{B}$. Assume that ∂Q intersects $\gamma \cup a$ minimally. Suppose also that $|\partial Q \cap a| > 0$ and that no component of Q is a sphere or disc disjoint from $a \cup \gamma$. We think of Q as being $\overline{Q} \cap N$ where $\overline{Q}$ is a surface in the manifold obtained from attaching 2–handles along the curves of $\mathscr{B}$ and filling in any 2–sphere boundary components with 3–balls. $\overline{Q}$ is obtained from Q by attaching discs to the components of ∂Q parallel to curves in $\mathscr{B}$. We then have $\partial_0 Q = \partial \overline{Q}$. Define $\Delta_i = |b_i \cap a|$, $\Delta_\partial = |\partial_0 Q \cap a|$, $\nu_i = |b_i \cap \gamma|$, $\nu_\partial = |\partial_0 Q \cap \gamma|$ and

$$K(\overline{Q}) = \sum_{i=1}^{|\mathscr{B}|} q_i(\Delta_i - \nu_i - 2) + (\Delta_\partial - \nu_\partial).$$

3.1. The first sutured manifold theorem

We begin with a definition.

DEFINITION. An a–boundary compressing disc for Q is a boundary compressing disc D with ∂D consisting of two arcs $\delta \cup \varepsilon$ so that $\delta \cap \varepsilon = \partial\delta = \partial\varepsilon$ and δ is an essential arc in Q and ε is a subarc of some essential simple closed curve in $\eta(a) \subset F$.

EXAMPLE. See Figure 3.1. In that figure, we are looking down $\overline{\alpha}$, the cocore of the 2–handle α. The parameterizing surface Q runs along $\overline{\alpha}$ twice, that is $|\partial Q \cap a| = 2$. An a–boundary compressing disc D for Q is shown at the far end of $\overline{\alpha}$.

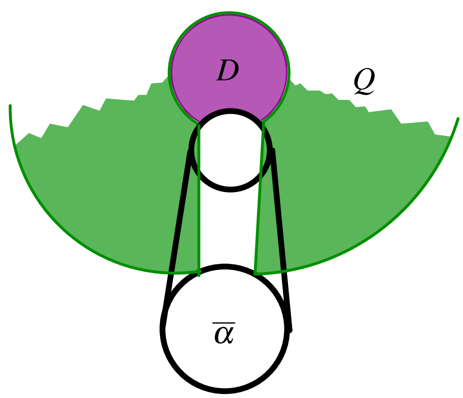

FIGURE 3.1. An a–boundary compression.

THEOREM 3.1. *Let $(N, \gamma \cup a)$ and Q be as described above. If either of the following holds:*

- *$(N[a], \gamma)$ is not taut*
- *There exists a surface S in $N[a]$ which is disjoint from $\overline{\alpha}$, is a conditioned surface in $N[a]$, and is taut in N but is not taut in $N[a]$.*

then one of the following holds:

(1) $N[a]$ *contains an essential separating sphere intersecting* $\overline{\alpha}$ *exactly twice and which cannot be isotoped to intersect that arc any fewer times. Furthermore, this sphere bounds a non-trivial homology ball in* $N[a]$.

(2) There is an a–boundary compressing disc for Q

(3) $-2\chi(\overline{Q}) \geq K(\overline{Q})$.

The remainder of this section proves the theorem. The proof was inspired by Lackenby's work [**L1**] on Dehn surgery on linking number zero knots in sutured manifolds.

We begin by creating a sequence of taut sutured manifold decompositions of $(N, \gamma \cup a)$. In order to effectively apply the main theorems of combinatorial sutured manifold theory, this sequence will need to be constructed in a particular fashion. The next lemma will provide the surfaces that are essential for creating a useful sutured manifold hierarchy.

LEMMA 3.2. *Suppose that* $(X, \Gamma \cup a)$ *is a taut sutured manifold and that* $H_2(X[a], \partial X[a]) \neq 0$. *Then, in* $X[a]$*, there is a conditioned surface* S *which is disjoint from* $\overline{\alpha}$. S *is a taut conditioned surface in* X *and* $\partial S \cap \eta(a) = 0$.

PROOF. By the proofs of Theorems 2.5 and 2.6 of [**S2**], given a non-trivial homology class $z \in H_2(X[a], \partial X[a])$ there exists a taut conditioned surface S' in the $\overline{\alpha}$–taut sutured manifold $(X[a], \Gamma, \overline{\alpha})$. After possibly replacing z with $-z$ we may assume that $\overline{\alpha}$ has algebraic intersection number

$i \geq 0$ with S'. By the choice of orientation of the arc $\overline{\alpha}$, $\overline{\alpha}$ has algebraic intersection number -1 with $R_+(\Gamma)$. The surface S'' which is the the double curve sum of S' with i copies of $R_+(\Gamma)$ has algebraic intersection number zero with $\overline{\alpha}$. Notice that $\partial S''$ satisfies the necessary criteria for S'' to be conditioned in $(X, \Gamma \cup a)$. Tube together points of opposite intersection number to create from S'' a surface S which is disjoint from $\overline{\alpha}$ and for which $\partial S = \partial S''$. The surface S is a conditioned surface in $(X, \Gamma \cup a)$. We may therefore replace S with a taut surface in $(X, \Gamma \cup a)$ having the same boundary. $\square$

To create a taut sutured manifold decomposition that is adapted to the parameterizing surface Q, we may need to take the double curve sum of our favorite conditioned surface S in a sutured manifold $(X, \Gamma \cup a)$ with some number k of copies of $R_+(\Gamma \cup a)$ and some number l of copies of $R_-(\Gamma \cup a)$, creating the surface $S_{k,l}$. We then decompose using the surface $S_{k,l}$ instead of S. The conditioned surfaces that we use will be the ones provided by Lemma 3.2. Performing the double curve sums creates boundary components of $S_{k,l}$ which are located in $\eta(a)$. Attaching discs to each of those boundary components creates a surface $S^a_{k,l} \subset X[a]$. The surface $S^a_{k,l}$ can also be created by taking the double curve sum of S with k copies of $R_+(\Gamma) \subset X[a]$ and l copies of $R_-(\Gamma) \subset X[a]$.

The next lemma guarantees that if we use such a surface to perform a decomposition of the sutured manifold $(X[a], \Gamma, \overline{\alpha})$ then all but one arc of $\overline{\alpha} - \eta(S^a_{k,l})$ can be cancelled. Let $*_a$ denote the point on $\overline{\alpha}$ to which the curve a retracts under the standard retraction of $\eta(\overline{\alpha})$ to $\overline{\alpha}$. If $(X[a], \Gamma, \overline{\alpha})$

is a sutured manifold which is decomposed along a surface Σ which is disjoint from $*_a$, call any component of $\overline{\alpha} - \mathring{\eta}(\Sigma)$ which doesn't contain $*_a$ a **residual arc**. If such an arc is converted into a suture (Lemma 2.5), call the resulting suture a **residual suture**.

LEMMA 3.3. *Suppose $S^a_{k,l} \subset X[a]$ be a surface created from the S provided by Lemma 3.2. Then after decomposing $X[a]$ along $S^a_{k,l}$ there exists a cancelling disc or a self-amalgamating disc for each residual arc.*

PROOF. Suppose that $\overline{\alpha}'$ is a residual arc. Since S is disjoint from a, each endpoint of $\overline{\alpha}'$ is on $R_\pm(\Gamma)$ or on a pushed off copy of $R_\pm(\Gamma)$. In other words, since $\overline{\alpha}'$ is a residual arc both endpoints are on different copies of $R_+(\Gamma)$ or on different copies of $R_-(\Gamma)$. Let P be the product region $R_\pm \times I$ between these copies. Then each component of $S \cap P$ is an (arc) $\times I$ or an $S^1 \times I$. The arc $\overline{\alpha}' = \overline{\alpha} \cap P$ is a copy of (point) $\times I$.

If the component of $(R_\pm \times \{0\}) - S$ containing the endpoint of $\overline{\alpha}'$ has any part of its boundary intersecting $\partial(R_\pm \times \{0\})$ choose a path p in that component from $\partial\overline{\alpha}'$ to $\partial(R_\pm \times \{0\})$. If not, then there is an essential closed curve p in that component which passes through the endpoint of $\overline{\alpha}'$ and is isotopic to a component of ∂S. In the first case, $p \times I$ is a cancelling disc for $\overline{\alpha}'$ and in the second case, $p \times I$ is a self-amalgamating disc for $\overline{\alpha}'$. □

In creating a hierarchy of $(N, \gamma \cup a)$ it may be necessary to eliminate index zero discs. Certain index-zero discs need to be treated carefully. To that end, suppose that $(X[a], \Gamma, \overline{\alpha})$ is a sutured manifold with $\overline{\alpha}$ a collection of arcs. Suppose that $D \subset X[a]$ is a cancelling disc for a component $\overline{\alpha}'$ of $\overline{\alpha}$. A regular neighborhood of D is a 3–ball B containing $\overline{\alpha}'$. Cutting open $X[a]$ along the disc $E = \mathrm{cl}(\partial B \cap \mathring{X})$ produces a sutured manifold, one component of which is a 3–ball containing $\overline{\alpha}'$. The 3–ball has a single suture in its boundary. We may then remove the arc $\overline{\alpha}'$ without affecting $\overline{\alpha}$-tautness. The other component is the sutured manifold we would obtain by cancelling the arc $\overline{\alpha}'$ in $X[a]$. By converting all arcs to sutures we obtain a decomposition of $(X, \Gamma \cup a)$ which eliminates the index zero disc D. Indeed, by decomposing along E but not the disc D we can eliminate an index-zero disc in X without cutting along a or a residual suture. This is at the cost of introducing a component which is a solid torus having two longitudinal sutures in its boundary. Exactly one of those sutures is either a or a residual suture. If it is a residual suture call the component a **residual torus**.

Suppose that $D \subset X[a]$ is a self-amalgamating disc for a component $\overline{\alpha}'$ of $\overline{\alpha}$. Then slightly enlarging it produces a non-trivial product annulus A in X. There is a parallelism of $\overline{\alpha}$ in $X[a]$ into A. After decomposing $X[a]$ along A there is a cancelling disc for $\overline{\alpha}'$ which may then be eliminated as above. Notice that since each component of ∂S is essential in $\partial X[a]$ (by the construction of S in Lemma 3.2) the product discs created by the self-amalgamating discs of Lemma 3.3 have ends which are essential in $\partial X[a]$. Thus, if a product annulus created from a self-amalgamating disc has both ends inessential in $\partial X[a]$ it must have arisen from a self-amalgamating disc for the suture a. But it is easy to see that in this case all such product annuli must have both ends essential in $\partial X[a]$. This observation will be useful in the proof of Lemma 3.5 below.

LEMMA 3.4. *There is a taut sequence of sutured manifold decompositions*

$$(\dagger)\qquad (N,\gamma\cup a)=(N_0,\gamma_0\cup a)\xrightarrow{S_1}(N_1,\gamma_1\cup a)\xrightarrow{S_2}\dots\xrightarrow{S_n}(N_n,\gamma_n\cup a)$$

adapted to the parameterizing surface Q such that

(1) *each decomposition is either a decomposition along an product disc or product annulus or along a surface $S_{k,l}$ given by Lemma 3.3. If the product disc intersects a residual suture then the decomposition is performed as described above. All decompositions along product annuli arise from this method of eliminating product discs, as described above.*
(2) $H_2(N_n[\mathfrak{a}],\partial N_n[\mathfrak{a}])=0$ *where $\mathfrak{a}$ is the curve a together with all the residual sutures.*
(3) *If a component of N_n does not contain a, it is either a residual torus or a 3–ball containing a single suture in its boundary.*

Another formulation of (2) is that if we convert a and all residual sutures to arcs, the resulting manifold has trivial homology relative to its boundary.

PROOF. This is essentially the proof that taut sutured manifold hierarchies exist (Theorem 2.2). The proof of that theorem makes the hierarchy stop when $H_2(N_n,\partial N_n)=0$. By Lemma 3.2, we can instead stop the hierarchy when $H_2(N_n[\mathfrak{a}],\partial N_n[\mathfrak{a}])=0$. If it is necessary to eliminate a product disc which intersects twice a residual suture or the suture a then the decomposition should be performed as described previously. By Lemma 3.3, there exists such a product disc for all residual sutures. Hence, all residual sutures end up in residual tori. Any component of N_n which does not contain a residual suture or a must be a 3–ball with a single suture in its boundary since $(N_n, \gamma_n \cup a)$ is taut and $H_2(N_n[\mathfrak{a}], \partial N_n[\mathfrak{a}]) = 0$. $\square$

Let N' denote the component of N_n which contains a. We can now use the hypotheses of the theorem we are trying to prove to conclude that $(N'[a], \gamma_n \cap N')$ is not taut.

LEMMA 3.5. *$(N'[a], \gamma_n \cap N')$ is not taut.*

PROOF. Since a component of $N_n - N'$ is either a 3–ball with a single suture in its boundary or a residual torus, all components of $N_n[\mathfrak{a}] - N'[a]$ are $\varnothing$–taut. Thus, if $(N'[a], \gamma_n \cap N')$ is taut, so is $(N_n[\mathfrak{a}], \gamma_n - \mathfrak{a})$.

Convert the hierarchy (†) into a sequence of sutured manifold decompositions of the sutured manifold $(N[a], \gamma, \overline{\alpha})$ by converting the suture a into an arc $\overline{\alpha}$ and any surface $S_{k,l}$ into $S^a_{k,l}$ as described previously. Let S^a_1 denote the result of applying this conversion to S_1. Then each surface in the hierarchy is either a product disc, non-trivial product annulus (by the remarks preceding Lemma 3.4), or conditioned surface. Thus, if $(N_n[\mathfrak{a}], \gamma_n - \mathfrak{a})$ is $\varnothing$-taut, by Theorem 2.3, $(N[a], \gamma)$ is taut and S^a_1 is taut. The surface S^a_1 is obtained by taking the double curve sum of S with k copies of $R_+(\gamma)$ and l copies of $R_-(\gamma)$. If S is not $\varnothing$–taut, then it does not minimize the Thurston norm (in $H_2(N[a], \partial S)$). But in this case, the double curve sum of S with k copies of R_+ and l copies of R_- is not Thurston norm minimizing either, implying that S^a_1 is not taut, a contradiction. Thus, if $(N'[a], \gamma_n \cap N')$ is taut so is $(N_n[\mathfrak{a}], \gamma_n - \mathfrak{a})$. In which case, we can also conclude that $(N[a], \gamma)$ and S are taut. But this contradicts the hypotheses of our theorem. $\square$

REMARK. Here is a brief aside to explain the route taken for the proof up until this point. Psychologically, it would be easier to have taken an $\overline{\alpha}$–taut hierarchy of $(N[a], \gamma)$. However, would need that at the end of the hierarchy there is at most one arc which cannot be cancelled. This requires that the conditioned surfaces be taken to be disjoint from $\overline{\alpha}$ (except for the result of double curve summing with $R_{\pm}$. *A priori* decompositions along such surfaces may not be $\overline{\alpha}$–taut. There is then no clear way to guarantee that the sutured manifold at the end is $\overline{\alpha}$–taut, in other words that it has the structure that we will now be making use of. Furthermore, it is unclear whether or not such a sequence of decompositions can be guaranteed to terminate. The proof given here avoids these difficulties by constructing taut decompositions of $(N, \gamma \cup a)$.

Carefully examining N' will enable us to conclude the proof of the theorem.

LEMMA 3.6. *$\partial N'$ is a torus and $N'[a]$ is an integer homology ball.*

PROOF. The proof is similar to [**L2**, Lemma A.4]. Let $A = \partial N' - \mathring{\eta}(a)$. By construction of the hierarchy, $H_2(N', A) = 0$. Thus, by duality for manifolds with boundary $H^1(N', \eta(a)) = 0$. By the Universal Coefficient Theorem, $H_1(N', \eta(a)) = 0$. From the exact sequence for the homology of the pair $(N', \eta(a))$, $H_1(\eta(a))$ surjects onto $H_1(N')$. Thus, $H_1(N')$ is cyclic. Since $H_2(N', A) = 0$, by the long exact sequence for the pair (N', A), $H_1(A)$ injects into $H_1(N')$. Since A is a surface and $\partial\eta(a)$ has two components, A is a collection of spheres and either an annulus or two discs. Since a does not compress in N, A does not contain a disc. The existence of a sphere would contradict tautness of N', and so A is an annulus.

Since $H_1(A)$ is isomorphic to $\mathbb{Z}$ and it injects into the cyclic group $H_1(N')$, $H_1(N')$ is also isomorphic to $\mathbb{Z}$. Since $\eta(a)$ is an annulus and since $H_1(\eta(a))$ surjects $H_1(N')$, the inclusion of $\eta(a)$ into N' induces an isomorphism on first homology. Since A is an annulus and $H_2(N',A) = 0$, the exact sequence for the pair (N',A) shows that $H_2(N') = 0$. It is then easy to see that $N'[a]$ is a homology ball. $\square$

Since $(N', (\gamma_n \cup a) \cap N')$ is a sutured manifold and $\partial N'$ is a torus containing the suture a there must be an odd number r of other sutures. The proof of the theorem concludes by examining two cases. The first case is when $r = 1$ and the second case is when $r \geq 3$.

Suppose that $r = 1$. Then $\partial N'[a]$ is a sphere containing a single suture. Since $(N'[a], \gamma_n \cap N')$ is an integer homology ball (Lemma 3.6) which is not taut, the integer homology ball is not a 3–ball. Push $\partial N'[a]$ slightly into $N[a]$. Then, $\partial N'[a]$ must be a reducing sphere for $N[a]$ which is intersected exactly twice by $\overline{\alpha}$ and which bounds a non-trivial integer homology ball. If $\overline{\alpha}$ could be isotoped to intersect the sphere $\partial N'[a]$ fewer times, it could be isotoped to be disjoint from that sphere and N' would be reducible, contrary to the hypothesis that $(N, \gamma \cup a)$ is taut. Hence, conclusion (1) holds.

Suppose, therefore, that $r \geq 3$. Let Q_n be the parameterizing surface in N_n obtained from Q. Since index does not increase during a hierarchy Theorem 2.4, the index of Q_n is no more than the index of Q. No component of Q_n is a sphere or a disc disjoint from γ_n, hence each component of Q_n has non-negative index. Suppose that ζ is a component of ∂Q_n which crosses a at least once. Let $A = \partial N' - \mathring{\eta}(a)$. If $\zeta \cap A$ contains an arc inessential in A

then either there is an isotopy of Q reducing $|\partial Q \cap \eta(a)|$ or an outermost such arc in A bounds an a–boundary compressing disc D for Q in N. The former is forbidden by our hypothesis that ∂Q intersects $\eta(a)$ minimally and the latter is the second of our conclusions. We may, therefore, assume that ζ is an essential loop in the torus $\partial N'$ which intersects $\eta(a)$ minimally a positive number of times. Hence, ζ intersects all $r+1$ sutures on $\partial N'$.

Let Q' be a component of Q_n such that at least one component of $\partial Q'$ intersects $\eta(a)$. Notice that $-2\chi(Q') \geq -2$. Let $z_{Q'} = |\partial Q' \cap \eta(a)|$. Then $\partial Q'$ has at least $z_{Q'}(r+1)$ intersections with the sutures γ_n. Hence,

$$I(Q') \geq z_{Q'}(r+1) - 2\chi(Q') \geq z_{Q'}(r+1) - 2 \geq z_{Q'}(r-1).$$

Then,

$$I(Q_n) \geq \sum I(Q') \geq (r-1) \sum z_{Q'}$$

where the sums are taken over all components Q' of Q which have at least one boundary component intersecting $\eta(a)$. By the construction of Q_n from Q, we have that $\sum z_{Q'} = |\partial Q \cap a|$. Thus,

$$|\partial Q \cap \gamma| + |\partial Q \cap a| - 2\chi(Q) = I(Q) \geq I(Q_n) \geq (r-1)|\partial Q \cap a|.$$

Consequently,

$$|\partial Q \cap \gamma| - 2\chi(Q) \geq (r-2)|\partial Q \cap a| \geq |\partial Q \cap a|$$

since $r \geq 3$.

Recalling that Q is obtained from $\overline{Q}$ by removing q_i discs with boundary parallel to b_i, that $|\partial Q \cap \gamma| = \sum q_i \nu_i + \nu_\partial$, that $|\partial Q \cap a| = \sum q_i \Delta_i + \Delta_\partial$, and that $\chi(Q) = \chi(\overline{Q}) - \sum q_i$ we obtain:

$$\sum q_i \nu_i + \nu_\partial - 2\chi(\overline{Q}) + 2\sum q_i \geq \sum q_i \Delta_i + \Delta_\partial.$$

It is easy to rearrange this to obtain

$$-2\chi(\overline{Q}) \geq K(\overline{Q})$$

as desired. □

3.2. The second sutured manifold theorem

In this section we show how to extend the argument of Theorem 9.1 of [**S3**] to allow non-planar parameterizing surfaces Q and how to replace the knot in that theorem with the arc $\overline{\alpha}$.

DEFINITION. An a–torsion $2g$–gon is a disc $D \subset N - \mathring{\eta}(Q)$ such that ∂D is divided into $2g$ subarcs $\delta_1, \varepsilon_1, \ldots \delta_g, \varepsilon_g$. Each subarc δ_i is an essential arc in Q. The subarcs ε_i are mutually parallel arcs in $\eta(a) - \partial Q$ all with the same orientation and all subarcs of essential circles in $\eta(a)$. Since they are mutually parallel they are contained in a rectangle $R \subset (F - \partial Q)$, with two edges of R subarcs of ∂Q. We require that the surface obtained by attaching R to Q be orientable.

EXAMPLE. Figure 3.2 shows a hypothetical example. The surface outlined with dashed lines is Q. It has boundary components on F. There are two such boundary components pictured. The curve running through Q and F

could be the boundary of an a–torsion 4–gon. Notice that the arcs ε_1 and ε_2 are parallel and oriented in the same direction. Attaching the rectangle containing those arcs as two of its edges to Q produces an orientable surface.

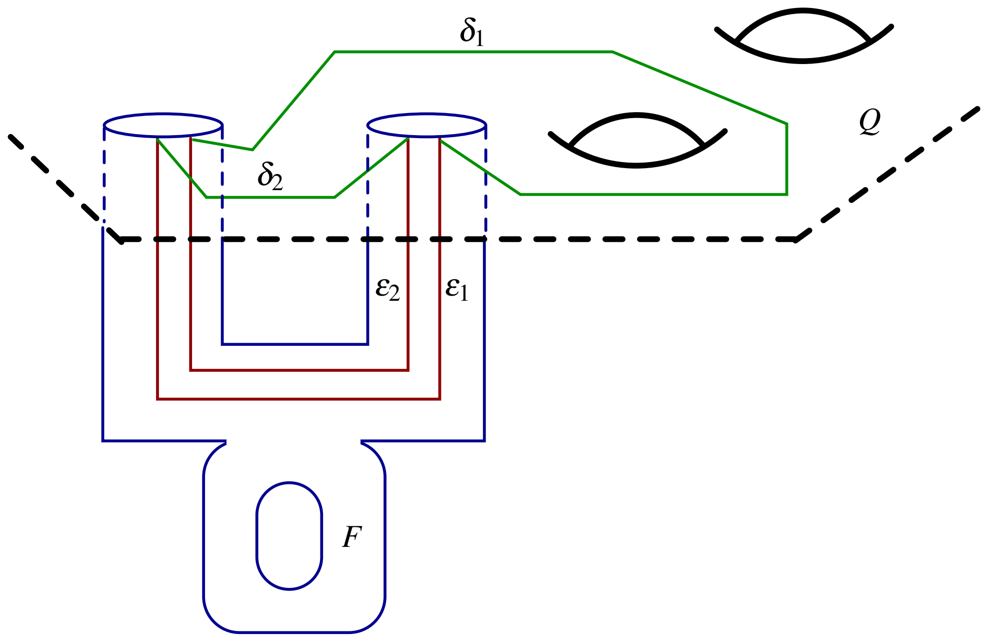

FIGURE 3.2. The boundary of an a–torsion 4–gon.

REMARK. The reason for the name a–torsion $2g$–gon will be clear in Section 5. In that section $|\mathscr{B}| \leq 2$. It will be shown that if $\overline{Q}$ is a sphere or disc and there is an a–torsion $2g$–gon for $Q = \overline{Q} \cap N$ with $g \geq 2$ then $H_1(N[b_1])$ is not torsion-free (in fact, $N[b_1]$ will contain a lens space summand). In general, however, the existence of an a–torsion $2g$–gon does not guarantee that $H_1(N[b_1])$ has torsion.

Notice that an a–torsion 2–gon is an a–boundary compressing disc. The main result of this section is similar to the first sutured manifold theorem except that instead of considering conditioned taut surfaces in N we consider conditioned $\overline{\alpha}$–taut surfaces in $N[a]$. The possible existence of an a–torsion $2g$–gon is weaker then the corresponding conclusion in the first sutured manifold theorem. We do not need to worry about an essential sphere in $N[a]$ intersecting $\overline{\alpha}$ twice, but we do need to worry that $N[a]$ may have torsion in first homology.

THEOREM 3.7 (cf. [**S3**, Theorem 9.1] and [**S4**, Proposition 4.1]). *Suppose that $(N[a], \gamma)$ is $\overline{\alpha}$–taut and that either*

- *$N[a]$ is not $\varnothing$–taut*
- *there is a conditioned $\overline{\alpha}$–taut surface $S \subset N[a]$ which is not $\varnothing$–taut.*
- *$N[a]$ is homeomorphic to a solid torus $S^1 \times D^2$ and $\overline{\alpha}$ cannot be isotoped so that its projection to the S^1 factor is monotonic.*

Then at least one of the following holds:

- *There is an a–torsion $2g$–gon for Q for some $g \in \mathbb{N}$*
- *$H_1(N[a])$ contains non-trivial torsion*
- $-2\chi(\overline{Q}) \geq K(\overline{Q})$.

REMARK. If $\overline{\alpha}$ can be isotoped to be monotonic in the solid torus $N[a]$ then it is, informally, a "braided arc". The contrapositive of this aspect of the theorem is similar to the conclusion in [**G2**] and [**S4**] that if a non-trivial surgery on a knot with non-zero wrapping number in a solid torus produces a solid torus then the knot is a 0 or 1-bridge braid.

The remainder of this section proves the theorem. Following [**S4**], define a **Gabai disc** for Q to be an embedded disc $D \subset N[a]$ such that

- $|\overline{\alpha} \cap \mathring{D}| > 0$ and all points of intersection have the same sign of intersection
- $|Q \cap \partial D| < |\partial Q \cap \eta(a)|$

The next proposition points out that the existence of a Gabai disc guarantees the existence of an a–boundary compressing disc or an a–torsion $2g$–gon.

PROPOSITION 3.8. *If there is a Gabai disc for Q then there is an a–torsion $2g$–gon.*

PROOF. Let D be a Gabai disc for Q. The intersection of Q with D produces a graph Λ on D. The vertices of Λ are ∂D and the points $\overline{\alpha} \cap D$. The latter are called the **interior vertices** of Λ. The edges of Λ are the arcs $Q \cap D$. A **loop** is an edge in Λ with initial and terminal points at the same vertex. A loop is **trivial** if it bounds a disc in D with interior disjoint from Λ.

To show that there is an a–torsion $2g$–gon for Q, we will show that the graph Λ contains a "Scharlemann cycle" of length g. The interior of the Scharlemann cycle will be the a–torsion $2g$–gon. In our situation, Scharlemann cycles will arise from a labelling of Λ which is slightly non-standard. Traditionally, when $\overline{\alpha}$ is a knot instead of an arc, the labels on the endpoints of edges in Λ, which are used to define "Scharlemann cycles", are exactly the components of ∂Q. In our case, since each component of ∂Q likely intersects $\partial \alpha$ more than once we need to use a slightly different labelling. After defining the labelling and the revised notion of "Scharlemann cycle", it will be clear to those familiar with the traditional situation that the new Scharlemann cycles give rise to the same types of topological conclusions as in the traditional setting. The discussion is modelled on Section 2.6 of [**CGLS**].

A Scharlemann cycle of length 1 is defined to be a trivial loop at an interior vertex of Λ. We now work toward a definition of Scharlemann cycles of length $g > 1$. Without loss of generality, we may assume that $|\overline{\alpha} \cap D| \geq 2$. Recall that the arc $\overline{\alpha}$ always intersects the disc D with the same sign. There is, in F, a regular neighborhood A of a such that $D \cap F \subset A$. We may choose A so that $\partial A \subset D \cap F$. Let $\partial_{\pm} A$ be the two boundary components of A. The boundary components of Q all have orientations arising from the orientation of $\overline{Q}$ and $\overline{\beta}$. We may assume by an isotopy that all the arcs $\partial Q \cap A$ are fibers in the product structure on A. Cyclically around A label the arcs of $\partial Q \cap A$ with labels $c_1 \ldots c_{\mu}$. Let $\mathscr{C}$ be the set of labels. Being a submanifold of ∂Q, each arc is oriented. Say that two arcs are **parallel** if they run through A in the same direction (that is, both from $\partial_- A$ to $\partial_+ A$ or both from $\partial_+ A$ to $\partial_- A$). Call two arcs **antiparallel** if they run through A in opposite directions. Note that since the orientations of $\mathring{D} \cap \partial W$ in A are all the same, an arc intersects each component of $\mathring{D} \cap \partial W$ with the same algebraic sign.

Call an edge of Λ with at least one endpoint on ∂D a **boundary edge** and call all other edges **interior edges**. As each edge of Λ is an arc and as all vertices of Λ are parallel oriented curves on ∂W, an edge of Λ must have endpoints on arcs of $\mathscr{C} = \{c_1, \ldots, c_{\mu}\}$ which are antiparallel. We call this the **parity principle** (as in [**CGLS**]). Label each endpoint of an edge in Λ with the arc in $\mathscr{C}$ on which the endpoint lies.

We will occasionally orient an edge e of Λ; in which case, let $\partial_- e$ be the tail and $\partial_+ e$ the head. A **cycle** in Λ is a subgraph homeomorphic to a circle. An x–cycle is a cycle which, when each edge e in the cycle is given a consistent orientation, has $\partial_- e$ labelled with $x \in \mathscr{C}$. Let Λ' be a subgraph of Λ and let x be a label in $\mathscr{C}$. We say that Λ' satisfies condition $P(x)$ if:

$P(x)$: For each vertex v of Λ' there exists an edge of Λ' incident to v with label x connecting v to an interior vertex.

LEMMA 3.9 ([**CGLS**, Lemma 2.6.1]). *Suppose that Λ' satisfies $P(x)$. Then each component of Λ contains an x–cycle.*

PROOF. The proof is the same as in [**CGLS**]. □

A **Scharlemann cycle** is an x–cycle σ where the interior of the disc in D bounded by σ is disjoint from Λ. See Figure 3.3. Since each intersection point of $D \cap \overline{\alpha}$ has the same sign, the set of labels on a Scharlemann cycle contains x and precisely one other label y, a component of $\mathscr{C}$ adjacent to x in A. The arc y and the arc x are antiparallel by the parity principle. The **length** of the Scharlemann cycle is the number of edges in the x–cycle.

LEMMA 3.10 ([**CGLS**, Lemma 2.6.2]). *If Λ contains an x–cycle, then it contains a Scharlemann cycle.*

PROOF. The proof is again the same as in [**CGLS**]. □

REMARK. In [**CGLS**], there is a distinction between x–cycles and, so-called, great x–cycles. We do not need this here because all components of $D \cap F$ are parallel in $\eta(\partial\alpha)$ as oriented curves.

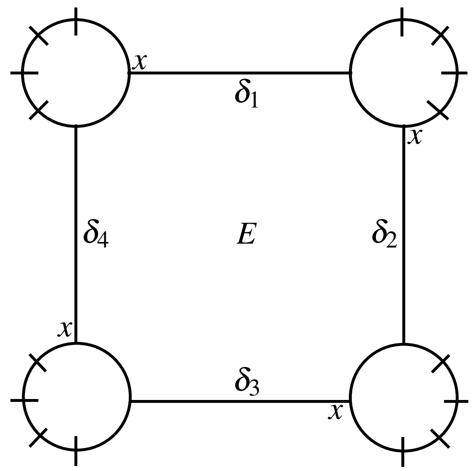

FIGURE 3.3. A Scharlemann cycle of length 4 bounding an a–torsion 8-gon.

The next corollary explains the necessity of considering Scharlemann cycles.

COROLLARY 3.11 ([**CGLS**]). *If ∂D intersects fewer than $|\partial Q \cap A|$ edges of Λ then Λ contains a Scharlemann cycle.*

PROOF. As ∂D contains fewer than $|\partial Q \cap A|$ endpoints of boundary-edges in Λ there is some $x \in \mathscr{C}$ which does not appear as a label on a boundary edge. As every interior vertex of Λ contains an edge with label x at that vertex, none of those edges can be a boundary edge. Consequently, Λ satisfies $P(x)$. Hence, by Lemmas 3.9 and 3.10, Λ contains a Scharlemann cycle of length g (for some g). $\square$

In A there is a rectangle R with boundary consisting of the arcs x and y and subarcs of ∂A. See Figure 3.4. Because $\overline{\alpha}$ always intersects D with the same sign, ∂D always crosses R in the same direction. This shows that the arcs ε_i are all mutually parallel in F. The arcs x and y are antiparallel, so

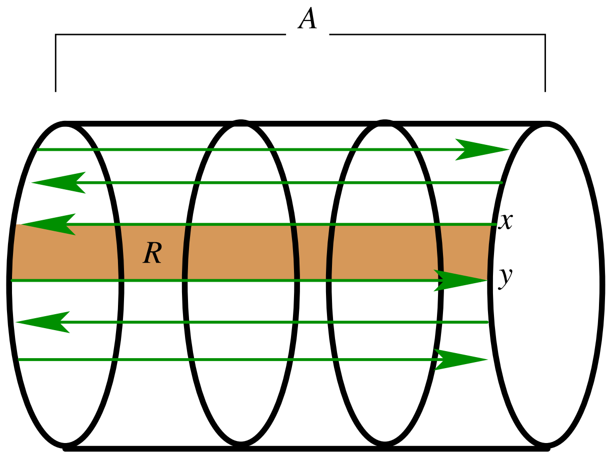

FIGURE 3.4. The rectangle R.

attaching R to Q produces an orientable surface. Hence, the interior of the Scharlemann cycle is an a–torsion $2g$–gon. □

We now proceed with proving the contrapositive of the theorem. Suppose that none of the three possible conclusions of the theorem hold. Let

$$(N[a],\gamma) \xrightarrow{S_1} (N_1,\gamma_1) \xrightarrow{S_2} \dots \xrightarrow{S_n} (N_n,\gamma_n)$$

be an $\overline{\alpha}$–taut sutured manifold hierarchy for $(N[a],\gamma)$ which is adapted to Q. The surface S_1 may be obtained from the surface S by performing the double-curve sum of S with k copies of R_+ and l copies of R_- (Theorem 2.1).

Since $-2\chi(\overline{Q}) < K(\overline{Q})$, simple arithmetic shows that $I(Q) < 2|\partial Q \cap \eta(a)|$. Since there is no a–torsion $2g$–gon for Q, by the previous proposition, there is no Gabai disc for Q. The proof of [**S3**, Theorem 9.1] shows that (N_n,γ_n) is also $\varnothing$–taut, after substituting the assumption that there are no Gabai discs for Q in N wherever [**S3**, Lemma 9.3] was used (as in [**S4**, Proposition

4.1]). In claims 3, 4, and 11 of [**S3**, Theorem 9.1] use the inequality $I(Q) < 2|\partial Q \cap A|$ to derive a contradiction rather than the inequalities stated in the proofs of those claims.

The sutured manifold hierarchy above is a sequence of sutured manifold decompositions satisfying the requirements of Theorem 2.3 (with empty 1–complex). Hence, the hierarchy is $\varnothing$–taut, $(N[a], \gamma)$ is a $\varnothing$–taut sutured manifold and S_1 is a $\varnothing$–taut surface. Suppose that S is not $\varnothing$–taut. Then there is a surface S' with the same boundary as S but with smaller Thurston norm. Then the double-curve sum of S' with k copies of R_+ and l copies of R_- has smaller Thurston norm than S_1, showing that S_1 is not $\varnothing$–taut. Hence, S is $\varnothing$–taut.

The proof of [**S3**, Theorem 9.1] concludes by noting that at the final stage of the hierarchy, there is a cancelling or (non-self) amalgamating disc for each remnant of $\overline{\alpha}$. When $N[a]$ is a solid torus the only $\varnothing$–taut conditioned surfaces are unions of discs. If S is chosen to be a single disc then S_1 is isotopic to S. To see this, notice that $R_\pm$ is an annulus and so the double-curve sum of S with $R_\pm$ is isotopic to S. Hence, the hierarchy has length one and the cancelling and (non-self) amalgamating discs show that $\overline{\alpha}$ is braided in $N[a]$. $\square$

CHAPTER 4

Placing Sutures

Let N be a compact, orientable, irreducible 3–manifold with $F \subset \partial N$ a component containing an essential simple closed curve a. Suppose that $\partial N - F$ is incompressible in N. For effective application of the first and second sutured manifold theorems, we need to choose curves γ on $\partial N[a]$ so that $(N[a], \gamma)$ is $\overline{\alpha}$–taut and $(N, \gamma \cup a)$ is $\varnothing$–taut. With our applications in mind, we restrict our attention to the situation when the boundary component F containing a has genus 2. Define $\partial_1 N[a] = \partial N - F$ and $\partial_0 N[a] = \partial N[a] - \partial_1 N[a]$.

For the moment, we consider only the choice of sutures $\hat{\gamma}$ on $\partial_0 N[a]$. If a is separating, so that $\partial_0 N[a]$ consists of two tori joined by the arc $\overline{\alpha}$, we do not place any sutures on $\partial_0 N[a]$, i.e. $\hat{\gamma} = \varnothing$. (Figure 4.1.A.) If a is non-separating, choose $\hat{\gamma}$ to be a pair of disjoint parallel loops on $F - \eta(a)$ which separate the endpoints of $\overline{\alpha}$. (Figure 4.1.B.)

If we are in the special situation of "refilling meridians", we will want to choose the curves $\hat{\gamma}$ more carefully. Recall that in this case $N \subset M$ and F bounds a genus 2 handlebody $W \subset (M - \mathring{N})$. The curves a and b bound in W discs α and β respectively.

Assuming that the discs β and α have been isotoped to intersect minimally and non-trivially the intersection $\alpha \cap \beta$ is a collection of arcs. An arc of

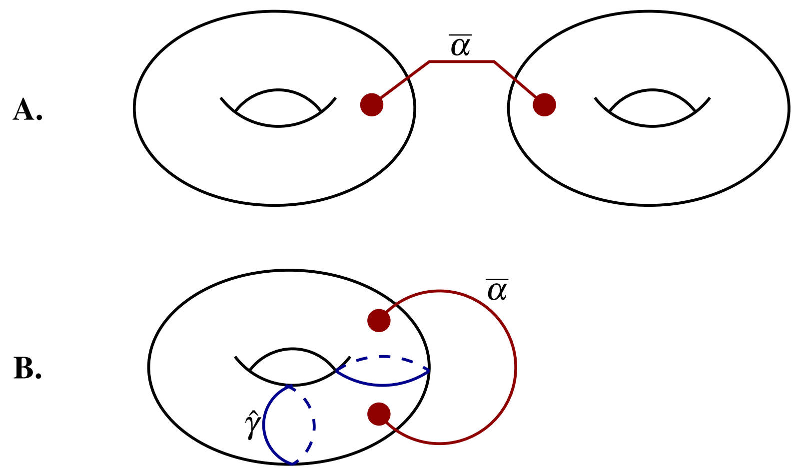

FIGURE 4.1. Choosing $\hat{\gamma}$.

$\alpha \cap \beta$ which is outermost on β cobounds with a subarc ψ of b a disc with interior disjoint from α. This disc is a meridional disc of a (solid torus) component of $\partial W - \mathring{\eta}(\alpha)$. The arc ψ has both endpoints on the same component of $\partial\eta(a) \subset F$. We, therefore, define a **meridional arc** of $b - a$ to be any arc of $b - \mathring{\eta}(a)$ which together with an arc in $\partial\eta(\alpha) \cap \mathring{W}$ bounds a meridional disc of $W - \mathring{\eta}(\alpha)$. If a is non-separating, then the existence of meridional arcs shows that every arc of $b - \mathring{\eta}(a)$ with endpoints on the same component of $\partial\eta(a) \subset F$ is a meridional arc of $b - a$. An easy counting argument shows that if a is non-separating then there are equal numbers of meridional arcs of $b - a$ based at each component of $\partial\eta(a) \subset F$. Hence, when a is non-separating, the number of meridional arcs of $b - a$, denoted $\mathscr{M}_a(b)$ is even. Some meridional arcs are depicted in Figure 4.2.

Returning to the definition of the sutures $\hat{\gamma}$, we insist that when "refilling meridians" and when α is non-separating, the curves $\hat{\gamma}$ be meridional curves

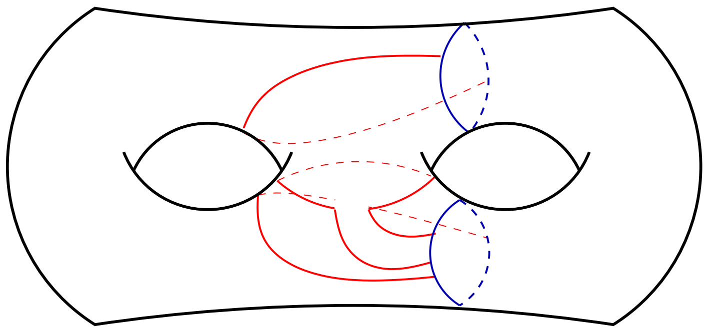

FIGURE 4.2. Some meridional arcs on ∂W

of the solid torus $W - \mathring{\eta}(\alpha)$ which separate the endpoints of $\overline{\alpha}$ and which are disjoint from the meridional arcs of $b - a$ for a specified b.

We now show how to define sutures $\widetilde{\gamma}$ on non-torus components of $\partial_1 N[a]$. Let $T(\gamma)$ be all the torus components of $\partial_1 N[a]$. If $\partial_1 N = T(\gamma)$ then $\widetilde{\gamma} = \varnothing$. Otherwise, the next lemma demonstrates how to choose $\widetilde{\gamma}$ so that, under certain hypotheses, $(N, \gamma \cup a)$ is taut, where $\gamma = \hat{\gamma} \cup \widetilde{\gamma}$.

LEMMA 4.1. *Suppose that $F - (\gamma \cup a)$ is incompressible in N. Suppose also that if $\partial_1 N[a] \neq T(\gamma)$ then there is no essential annulus in N with boundary on $\hat{\gamma} \cup a$. Then $\widetilde{\gamma}$ can be chosen so that $(N, \gamma \cup a)$ is $\varnothing$–taut and so that $(N[a], \gamma)$ is $\overline{\alpha}$–taut. Furthermore, if $c \subset \partial_1 N[a]$ is a collection of disjoint, non-parallel curves such that:*

- $|c| \leq 2$
- *All components of c are on the same component of $\partial_1 N[a]$*
- *No curve of c cobounds an essential annulus in N with a curve of $\hat{\gamma} \cup a$*
- *If $|c| = 2$ then there is no essential annulus in N with boundary c*

- *If $|c| = 2$ and a is separating, there is no essential thrice-punctured sphere in N with boundary $c \cup a$.*

then $\widetilde{\gamma}$ can be chosen to be disjoint from c.

The main ideas of the proof are contained in Section 5 of [**S4**] and Theorem 2.1 of [**L2**]. In [**S4**], Scharlemann considers "special" collections of curves on a non-torus component of ∂N. These curves cut the component into thrice-punctured spheres. Exactly two of the curves in the collection bound once-punctured tori. In those tori are two curves of the collection which are called "redundant". The redundant curves are removed and the remaining curves form the desired sutures. Scharlemann shows how to construct such a special collection which is disjoint from a set of given curves and which gives rise to a taut-sutured manifold structure on the manifold under consideration. Lackenby, in [**L2**], uses essentially the same construction (but with fewer initial hypotheses) to construct a collection of curves cutting the non-torus components of ∂N into thrice-punctured spheres, but where all the curves are non-separating. We need to allow separating curves in the sutures as c may contain separating curves. By slightly adapting Scharlemann's work, in the spirit of Lackenby, we can make do with the hypotheses of the lemma, which are slightly weaker than what a direct application of Scharlemann's work would allow.

PROOF. Let τ be the number of once-punctured tori in ∂N with boundary some component of $c \cup a$. Since all components of c are on the same component of ∂N, $\tau \leq 4$ with $\tau \geq 3$ only if a is separating.

Say that a collection of curves on ∂N is **pantsless** if, whenever a thrice-punctured sphere has its boundary a subset of the collection, all components of the boundary are on the same component of ∂N. If a is non-separating, then $\tau \leq 2$. Hence, either $\tau \leq 2$ or $c \cup a \cup \hat{\gamma}$ is pantsless.

Scharlemann shows how to extend the set c to a collection Γ, such that there is no essential annulus in N with boundary on $\Gamma \cup a \cup \hat{\gamma}$ and the curves Γ cut ∂N into tori, once-punctured tori, and thrice-punctured spheres. Furthermore, if $c \cup a \cup \hat{\gamma}$ is pantsless, then so is $\Gamma \cup a \cup \hat{\gamma}$. An examination of Scharlemann's construction shows that all curves of $\Gamma - c$ may be taken to be non-separating. Thus, the number of once-punctured tori in ∂N with boundary on some component of $\Gamma \cup a$ is still τ. If Γ cannot be taken to be a collection of sutures on ∂N, then, by construction, $|c| = 2$, one curve of c bounds a once-punctured torus in ∂N containing the other curve of c. The component of c in the once-punctured torus is "redundant" (in Scharlemann's terminology). If no curve of c is redundant, let $\widetilde{\gamma} = \Gamma$; otherwise, form $\widetilde{\gamma}$ by removing the redundant curve from Γ. Let $\gamma' = \widetilde{\gamma} \cup a \cup \hat{\gamma}$. We now have a sutured manifold (N, γ'). Notice that the number of once-punctured torus components of $\partial N - \gamma'$ is equal to τ.

We now desire to show that (N, γ') is $\varnothing$–taut. If it is not taut, then $R_{\pm}(\gamma)$ is not norm-minimizing in $H_2(N, \eta(\partial R_{\pm}))$. Let J be an essential surface in N with $\partial J = \partial R_{\pm} = \gamma'$. Notice that $\chi_{\varnothing}(R_{\pm}) = -\chi(\partial N)/2$ and that $|\gamma'| = -3\chi(\partial N)/2 - \tau$.

Recall that either $\tau \leq 2$ or γ' is pantsless. Suppose, first, that $\tau \leq 2$. Since no component of J can be an essential annulus, by the arguments of Scharlemann and Lackenby, $\chi_\varnothing(J) \geq |\partial J|/3 = |\gamma'|/3$. Hence,

$$\chi_\varnothing(J) \geq -\chi(\partial N)/2 - \tau/3.$$

Since $\tau \leq 2$ and since $\chi_\varnothing(J)$ and $-\chi(\partial N)/2$ are both integers, $\chi_\varnothing(J) \geq |\partial N|/2 = \chi_\varnothing(R_\pm)$. Thus, when $\tau \leq 2$, (N, γ') is a $\varnothing$–taut sutured manifold.

Suppose, therefore that γ' is pantsless. Recall that $\tau \leq 4$. We first examine the case when each component of J has its boundary contained on a single component of ∂M. Let J_0 be all the components of J with boundary on a single component T of ∂N. Let τ_0 be the number of once-punctured torus components of $T - \gamma'$. Notice that $\tau_0 \leq 2$. The proof for the case when $\tau \leq 2$, shows that $\chi_\varnothing(J_0) \geq \chi_\varnothing(R_\pm \cap T)$. Summing over all component of ∂N shows that $\chi_\varnothing(J) \geq \chi_\varnothing(R_\pm)$, as desired.

We may, therefore, assume that some component J_0 of J has boundary on at least two components of ∂N. Since γ' is pantsless, $\chi_\varnothing(J_0) \geq (|\partial J_0| + 2)/3$. For the other components of J we have, $\chi_\varnothing(J - J_0) \geq |\partial(J - J_0)|/3$. Thus,

$$\chi_\varnothing(J) \geq \frac{|\gamma'| + 2}{3} \geq -\frac{\chi(\partial N)}{2} + \frac{2 - \tau}{3}$$

Since $\tau \leq 4$ and since $\chi_\varnothing(J)$ and $-\chi(\partial N)/2$ are both integers, we must have $\chi_\varnothing(J) \geq -\chi(\partial N)/2 = \chi_\varnothing(R_\pm)$, as desired. Hence, $(N, \gamma') = (N, \gamma \cup a)$ is $\varnothing$–taut. Consequently, by Lemma 2.5, $(N[a], \gamma)$ is $\overline{\alpha}$–taut. $\square$

REMARK. The assumption that all components of c are contained on the same component of ∂M can be weakened to a hypothesis on the number τ. For what follows, however, our assumption suffices.

We will be interested in when a component of $\partial N - F$ becomes compressible upon attaching a 2–handle to $a \subset F$ and also becomes compressible upon attaching a 2–handle to $b \subset F$. If such occurs, the curves c of the previous lemma will be the boundaries of the compressing discs for that component of ∂N. Obviously, in order to apply the lemma we will need to make assumptions on how that component compresses.

CHAPTER 5

Constructing Q

The typical way in which we will apply the two main theorems is as follows. Suppose that a and b are simple closed curves on a genus two component $F \subset \partial N$ and that there is an "interesting" surface $\overline{R} \subset N[b]$. We will want to use this surface to show that either $-2\chi(\overline{R}) \geq K(\overline{R})$ or $N[a]$ is taut. *A priori*, though, the surface $R = \overline{R} \cap N$ may have a–boundary compressing discs or a–torsion $2g$–gons. The purpose of this section is to show how, given the surface $\overline{R}$ we can construct another surface $\overline{Q}$ which will, hopefully, have similar properties to $\overline{R}$ but be such that $Q = \overline{Q} \cap N$ does not have a–boundary compressing discs or a–torsion $2g$–gons. This goal will not be entirely achievable, but Theorem 5.1 shows how close we can come. Throughout we assume that N is a compact, orientable, irreducible 3–manifold with $F \subset \partial N$ a component having genus equal to 2. Let a and b be two essential simple closed curves on F so that a and b intersect minimally and non-trivially. As before, let $\partial_1 N = \partial_1 N[b] = \partial N - \partial F$ and let $\partial_0 N[b] = \partial N[b] - \partial_1 N[b]$. $\partial_0 N[b]$ has one or two components, depending on whether b is separating or non-separating. Let T_0 and T_1 denote these components, with $T_0 = T_1$ if b is non-separating.

Before stating the theorem, we make some important observations about $N[b]$ and surfaces in $N[b]$. If b is non-separating, there are multiple ways to obtain a manifold homeomorphic to $N[b]$. Certainly, attaching a 2–handle to b is one such way. If b^* is any curve in F which cobounds in F with $\partial\eta(b)$ a thrice-punctured sphere, then attaching 2–handles to both b^* and b creates a manifold with a spherical boundary component. Filling in that sphere with a 3–ball creates a manifold homeomorphic to $N[b]$. We will often think of $N[b]$ as obtained in this fashion. Say that a surface $\overline{Q} \subset N[b]$ is **suitably embedded** if each component of $\partial Q - \partial\overline{Q}$ is a curve parallel to b or to some b^*. We denote the number of components of $\partial Q - \partial\overline{Q}$ parallel to b by $q = q(\overline{Q})$ and the number parallel to b^* by $q^* = q^*(\overline{Q})$. If b is separating, define $b^* = \varnothing$. Let $\widetilde{q} = q + q^*$. Define $\Delta = |b \cap a|$, $\Delta^* = |b^* \cap a|$, $\nu = |b \cap \gamma|$, and $\nu^* = |b^* \cap \gamma|$. We then have

$$K(\overline{Q}) = (\Delta - \nu - 2)q + (\Delta^* - \nu^* - 2)q^* + \Delta_\partial - \nu_\partial.$$

Define a **slope** on a component of $\partial N[b]$ to be an isotopy class of pairwise disjoint, pairwise non-parallel curves on that component. The set of curves is allowed to be the empty set. Place a partial order on the set of slopes on a component of $\partial N[b]$ by declaring $r \leq s$ if there is some set of curves representing r which is contained in a set of curves representing s. Notice that $\varnothing \leq r$ for every slope r. Say that a surface $\overline{R} \subset N[b]$ has boundary slope $\varnothing$ on a component of ∂N if $\partial\overline{Q}$ is disjoint from that component. Say that a surface $\overline{R} \subset N[b]$ has boundary slope $r \neq \varnothing$ on a component of ∂N if each curve of $\partial\overline{R}$ on that component is contained in some representative of r and every curve of a representative of r is isotopic to some component of $\partial\overline{R}$.

Define a surface to be **essential** if it is incompressible, boundary-incompressible and has no component which is boundary-parallel or which is a 2–sphere bounding a 3–ball. The next theorem takes as input an essential surface $\overline{R} \subset N[b]$ and gives as output a surface $\overline{Q}$ such that $Q = \overline{Q} \cap N$ can (in many circumstances) be effectively used as a parameterizing surface in the first and second sutured manifold theorems. The remainder of the section will be spent proving it.

THEOREM 5.1. *Suppose that $\overline{R} \subset N[b]$ is a suitably embedded essential surface and suppose either*

(I) *$\overline{R}$ is a collection of essential spheres and discs, or*

(II) *$N[b]$ contains no essential sphere or disc.*

Then there is a suitably embedded incompressible and boundary-incompressible surface $\overline{Q} \subset N[b]$ with the following properties. (The properties have been organized for convenience. The properties marked with a "" are optional and need not be invoked.)*

- *$\overline{Q}$ is no more complicated than $\overline{R}$:*
  - (C1) $(-\chi(\overline{Q}), \widetilde{q}(\overline{Q})) \leq (-\chi(\overline{R}), \widetilde{q}(\overline{R}))$ *in lexicographic order*
  - (C2) *The sum of the genera components of $\overline{Q}$ is no bigger than the sum of the genera of components of $\overline{R}$*
  - (C3) *$\overline{Q}$ and $\overline{R}$ represent the same class in $H_2(N[b], \partial N[b])$*

- *The options for a–boundary compressions and a–torsion $2g$–gons are limited:*
  - (D1) *Either there is no a–boundary compressing disc for Q or $\widetilde{q} =$ 0.*

(*D2) *If no component of $\overline{R}$ is separating and if $\widetilde{q} \neq 0$ then there is no a–torsion $2g$–gon for Q.*

(D3) *If $\overline{Q}$ is a disc or 2–sphere then either $N[b]$ has a lens space connected summand or there is no a–torsion $2g$–gon for Q with $g \geq 2$.*

(D4) *If $\overline{Q}$ is a planar surface then either there is no a–torsion $2g$–gon for Q with $g \geq 2$ or attaching 2–handles to $\partial N[b]$ along $\partial\overline{Q}$ produces a 3–manifold with a lens space connected summand.*

- *The boundaries are not unrelated:*

(*B1) *Suppose that (II) holds, that we are refilling meridians, that no component of $\overline{R}$ separates, and that $\partial\overline{R}$ has exactly one non-meridional component on each component of $\partial_0 N[b]$. Then $\overline{Q}$ has exactly one boundary component on each component of $\partial_0 N[b]$ and the slopes are the same as those of $\partial\overline{R} \cap \partial_0 N[b]$.*

(B2) *If $\partial\overline{R} \cap \partial_1 N$ is contained on torus components of $\partial_1 N$ or if neither (D2) or (B1) are invoked, then the boundary slope of $\overline{Q}$ on a component of $\partial_1 N[b]$ is less than or equal to the boundary slope of $\overline{R}$ on that component.*

(B3) *If (D2) is not invoked and if the boundary slope of $\overline{R}$ on a component of $\partial_0 N[b]$ is non-empty then the boundary slope of $\overline{Q}$ on that component is less than or equal to the boundary slope of $\overline{R}$.*

Property (B1), which is the most unpleasant to achieve, is present to guarantee that if $\overline{R}$ is a Seifert surface for L_β then $\overline{Q}$ (possibly after discarding components) is a Seifert surface for L_β. This is not used subsequently in this dissertation, but future work is planned which will make use of it. However, achieving property (D2) which is used here, requires similar considerations.

The only difficulty in proving the theorem is keeping track of the listed properties of $\overline{Q}$ and $\overline{R}$. Eliminating a–boundary compressions is psychologically easier than eliminating a–torsion $2g$–gons, so we first go through the argument that a surface $\overline{Q}$ exists which has all but properties (D2) - (D4). The argument may be easier to follow if, on a first reading, $\overline{R}$ is considered to be a sphere or essential disc. The proof is based on similar work in [**S5**], which restricts $\overline{R}$ to being a sphere or disc.

The main purpose of assumptions (I) and (II) is to easily guarantee that the process for creating $\overline{Q}$ described below terminates. We will show that if $\widetilde{q}(\overline{R}) \neq 0$ and there is an a–boundary compressing disc or a–torsion $2g$-gon for $R = \overline{R} \cap N$ then there is a sequence of operations on $\overline{R}$ each of which reduces a certain complexity but preserves the properties listed above (including essentiality of $\overline{R}$). If (I) holds, the complexity is simply $\widetilde{q}$. If (II) holds, the complexity is $(-\chi(\overline{R}), \widetilde{q}(\overline{R}))$ (with lexicographic ordering). If (II) holds, it is clear that $-\chi(\overline{R})$ is always non-negative. Thus each measure of complexity has a minimum. The process stops either when $\widetilde{q} = 0$ or when the minimum complexity is reached.

5.1. Eliminating a–boundary compressions

Assume that $\widetilde{q} \neq 0$ and that there is an a–boundary compressing disc D for R with $\partial D = \delta \cup \varepsilon$ where ε is a subarc of some essential circle in $\eta(a)$. There is no harm in considering $\varepsilon \subset a - \partial R$.

Case 1: b separates W. In this case, $\eta(\overline{\beta}) - \mathrm{int}\overline{R}$ consists $q-1$ copies of $D^2 \times I$ labelled $W_1, \ldots, W_{q-1}$. There are two components T_0 and T_1 of $\partial_0 N[b] = \partial N[b] - \partial N$, both tori. The frontiers of the W_j in $\eta(\overline{\beta})$ are discs $\beta_1, \ldots, \beta_q$, each parallel to β, the core of the 2–handle attached to b. Each 1-handle W_j lies between β_j and β_{j+1}. The torus T_0 is incident to β_1 and the torus T_1 is incident to β_q. See Figure 5.1.

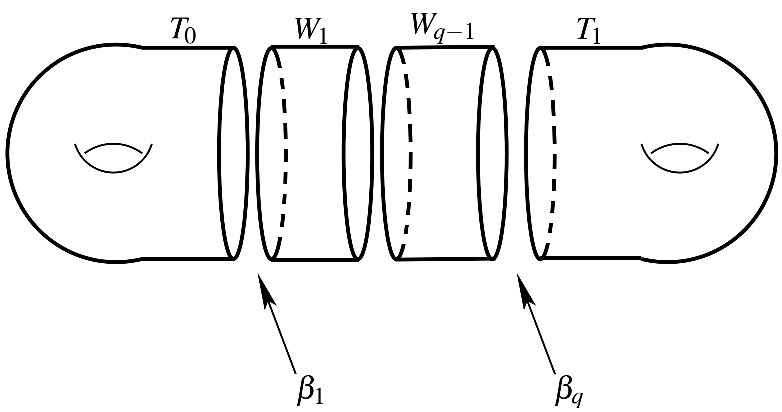

FIGURE 5.1. The tori and 1-handles W_j

The interior of the arc $\varepsilon \subset F$ is disjoint from ∂R. Consider the options for how ε could be positioned on W:

Case 1.1: ε lies in $\partial W_j \cap F$ for some $1 \leq j \leq q-1$. In this case, ε must span the annulus $\partial W_j \cap F$. The 1-handle W_j can be viewed as a regular neighborhood of the arc ε. The disc D can then be used to isotope W_j

through $\partial D \cap R$ reducing $|\overline{R} \cap \overline{\beta}|$ by 2. See Figure 5.2. This maneuver decreases $\widetilde{q}(\overline{R})$. Alternatively, the disc E describes an isotopy of $\overline{R}$ to a surface $\overline{Q}$ in $N[b]$ reducing $\widetilde{q}$. Clearly, $\overline{Q}$ satisfies the (C) and (B) properties.

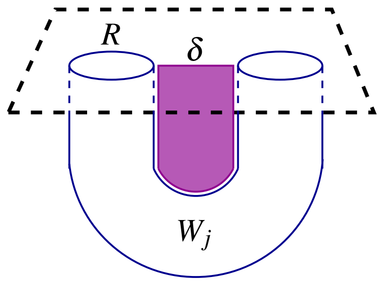

FIGURE 5.2. The disc D describes an isotopy of $\overline{R}$.

Suppose, then, that ε is an arc on T_0 or T_1. Without loss of generality, we may assume it is on T_0.

Case 1.2: ε lies in T_0 and has both endpoints on $\partial\overline{R}$. This is impossible since $\overline{R}$ was assumed to be essential in $N[b]$ and $\widetilde{q} > 0$.

Case 1.3: ε lies in T_0 and has one endpoint on $\partial\beta_1$ and the other on $\partial\overline{R}$. The disc D guides a proper isotopy of $\overline{R}$ to a surface $\overline{Q}$ in $N[b]$ which reduces $\widetilde{q}$. See Figure 5.3. Clearly, the (C) and (D) properties are satisfied.

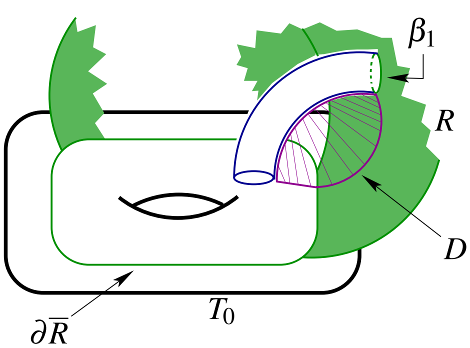

FIGURE 5.3. The disc D describes an isotopy of $\overline{R}$.

Case 1.4: ε lies in T_0 and has endpoints on $\partial\beta_1$. Boundary-compressing $\overline{R} - \mathring{\beta}_1$ produces a surface $\overline{J}$ with two new boundary components on T_0, both of which are essential curves. They are oppositely oriented and bound an annulus containing β_1. If $\partial\overline{R} \cap T_0 \neq \varnothing$ then these two new components have the same slope on T_0 as $\partial\overline{R}$, showing that property (B4) is satisfied. It is easy to check that $\chi(\overline{J}) = \chi(\overline{R})$ and that $\widetilde{q}(\overline{J}) = \widetilde{q}(\overline{R}) - 1$, so that (C1) is satisfied. Clearly, (C2), (C3), and (B3) are also satisfied.

If $\overline{J}$ were compressible, there would be a compressing disc for $\overline{R}$ by an outermost arc/innermost disc argument. Thus, $\overline{J}$ is incompressible. Suppose that E is a boundary-compressing disc for $\overline{J}$ in $N[b]$ with $\partial E = \kappa \cup \lambda$ where κ is an arc in $\partial N[b]$ and λ is an arc in $\overline{J}$. Since $\overline{R}$ is boundary-incompressible, the arc κ must lie on T_0 (and not on T_1). Since T_0 is a torus, either some component of $\overline{J}$ is a boundary-parallel annulus or $\overline{J}$ (and, therefore, $\overline{R}$) is compressible. We may assume the former. If $\overline{J}$ has other components apart from the boundary-parallel annulus, discarding the boundary-parallel annulus leaves a surface $\overline{Q}$ satisfying the (C) and (B) properties. We may, therefore, assume that $\overline{J}$ in its entirety is a boundary-parallel annulus.

Since $\chi(\overline{R}) = \chi(\overline{J})$, since $\overline{J}$ is a boundary-parallel annulus and since $\partial\overline{J}$ has two more components then $\partial\overline{R}$, $\overline{R}$ is an essential torus. However, using D to isotope $\eta(\delta) \subset \overline{R}$ into T_0 and then isotoping $\overline{J}$ into T_0 gives a homotopy of $\overline{R}$ into T_0, showing that it is not essential, a contradiction.

Thus, after possibly discarding a boundary-parallel annulus from $\overline{J}$ to obtain $\overline{L}$ we obtain a non-empty essential surface in $N[b]$ satisfying the first five required properties. If we do not desire property (B1) to be satisfied, take

$\overline{Q} = \overline{L}$. Notice that this step may, for example, convert an essential sphere into two discs or an essential disc with boundary on $\partial_1 N[b]$ into an annulus and a disc with boundary on $\partial_0 N[b]$. This fact accounts for the delicate phrasing of the (B) properties.

Suppose, therefore, that we wish to satisfy (B1). Among other properties, we assume that $\overline{R}$ has a single boundary component on T_0.

There is an annulus $A \subset T_0$ which is disjoint from $\beta_1 \subset T_0$, which has interior disjoint from $\partial\overline{L}$, and which has its boundary two of the two or three components of $\partial\overline{L}$. See Figure 5.4. In the figure, the dashed line represents the arc ε. The two circles formed by joining ε to $\partial\beta_1$ are the two new boundary components of $\overline{L}$. Since, they came from a boundary-compression, they are oppositely oriented. If $\partial\overline{R}$ has a single component on T_0 (indicated by the curve with arrows in the figure) then it must be oriented in the opposite direction from one of the new boundary components of $\partial\overline{L}$. Attaching A to $\overline{L}$ creates an orientable surface and does not increase negative euler characteristic or $\widetilde{q}$.

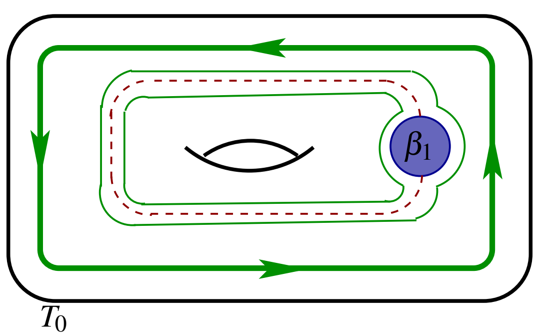

FIGURE 5.4. The annulus A lies between $\partial\overline{R}$ and one of the new boundary components of $\overline{L}$.

Thus, if $|\partial\overline{R}\cap T_0| \leq 1$, $\overline{L}\cup A$ is well-defined. It may, however, be compressible or boundary-compressible. Since it represents the homology class $[\overline{R}]$ in $H_2(N[b],\partial N[b])$, as long as that class is non-zero we may thoroughly compress and boundary-compress it, obtaining a surface $\overline{J}$. Discard all null-homologous components of $\overline{J}$ to obtain a surface $\overline{Q}$. By assumption (II), we never discard an essential sphere or disc. Notice that since $\partial\overline{R}$ has a single boundary component on T_1, the surface $\overline{Q}$ will also have a single boundary component on T_1. I.e. discarding separating components of $\overline{J}$ does not discard the component with boundary on T_1. Boundary-compressing $\overline{J}$ may change the slope of $\partial\overline{J}$ on non-torus components of $\partial_1 N[b]$. Discarding separating components may convert a slope on a torus component to the empty slope. Nevertheless, properties (B2) and (B3) still hold.

If a component of $\overline{J}$ is an inessential sphere then either $\overline{L}_A$ contained an inessential sphere or the sphere arose from compressions of $\overline{L}_A$. Suppose that the latter happened. Then after some compressions $\overline{L}_A$ contains a solid torus and compressing that torus creates a sphere component. Discarding the torus instead of the sphere shows that this process does not increase negative euler characteristic. If $\overline{L}_A$ contains an inessential sphere, this component is either a component of $\overline{L}$ and therefore of $\overline{R}$ or it arose by attaching A to two disc components, D_1 and D_2, of $\overline{L}$. The first is forbidden by the assumption that $\overline{R}$ is essential and the second by (II). Consequently, negative euler characteristic is not increased.

Notice that, in general, compressing $\overline{L}_A$ may increase $\widetilde{q}$, but because $-\chi(\overline{Q})$ is decreased, property (C1) is still preserved and complexity is decreased. Since we assume (II) for the maneuvre, if (I) holds at the end of this case we can still conclude that $\widetilde{q}$ was decreased. (This is an observation needed to show that the construction of $\overline{Q}$ for the conclusion of the theorem terminates.)

Case 2: b is non-separating and $q^* \neq 0$. This is very similar to Case 1. In what follows only the major differences are highlighted.

Since $q^* \neq 0$, the cocore $\overline{\beta}^*$ of the 2–handle attached to b^* and the cocore $\overline{\beta}$ form an arc with a loop at one end. Let $U = \eta(\overline{\beta^*} \cup \overline{\beta})$. Then $U - \overline{R}$ consists of a solid torus $q^* - 1$ copies of $D^2 \times I$ labelled $W_1^*, \ldots, W_{q^*-1}^*$ with frontier in U consisting of discs $\beta_1^*, \ldots, \beta_{q^*}^*$ parallel to β^* (the core of the 2–handle attached to b^*), a 3–ball $\mathscr{P}$ with frontier in U consisting of 3 discs: β_{q*}^*, β_1, and β_q, $q-1$ copies of $D^2 \times I$ labelled $W_1, \ldots, W_{q-1}$ with frontiers $\beta_1, \ldots, \beta_q$ consisting of discs parallel to β. See Figure 5.5. $\partial_0 N[b]$ consists of a single torus T_0.

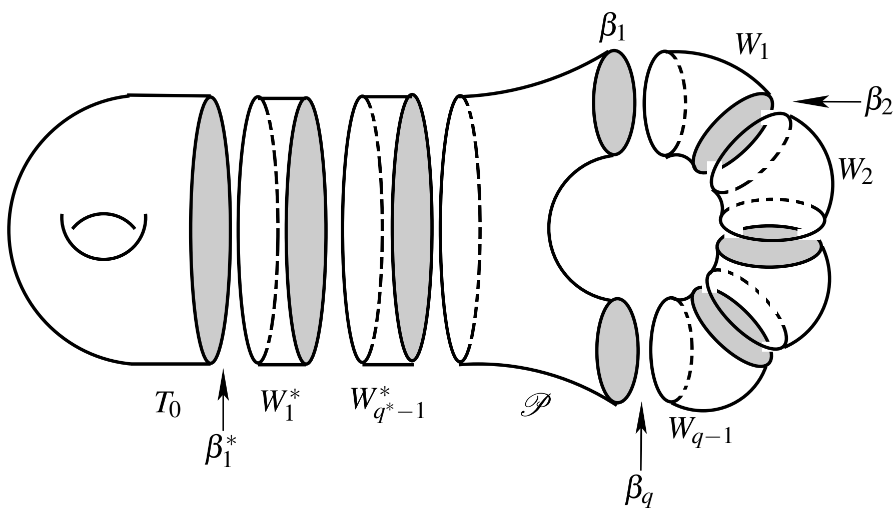

FIGURE 5.5. The torus, pair of pants, and 1-handles.

Case 2.1 : ε is not located in $\mathscr{P}$. This is nearly identical to Case 1. To achieve (B1), an "annulus attachment" trick like that in Case 1.4 is necessary.

Case 2.2: ε is located in $\mathscr{P}$. Since $\partial\overline{R}$ is essential in $N[b]$ and since $\overline{R}$ is embedded, $\partial\overline{R}$ is disjoint from $\mathscr{P}$. The arc ε has its endpoints on exactly two of $\{\partial\beta^*_{q^*}, \partial\beta_1, \partial\beta_q\}$. Denote by x and y the two discs containing $\partial\varepsilon$ and denote the third by z. That is, $\{\partial x, \partial y, \partial z\} = \{\partial\beta^*_{q^*}, \partial\beta_1, \partial\beta_q\}$. Boundary-compressing $\mathrm{cl}(\overline{Q} - (x \cup y))$ along D removes ∂x and ∂y as boundary components of R and adds another boundary-component parallel to ∂z. Attach a disc in F parallel to z to this new component, forming $\overline{J}$. $\overline{J}$ is isotopic in $N[b]$ to $\overline{R}$ (Figure 5.6) and is, therefore, essential and satisfies the (C) and (B) properties.

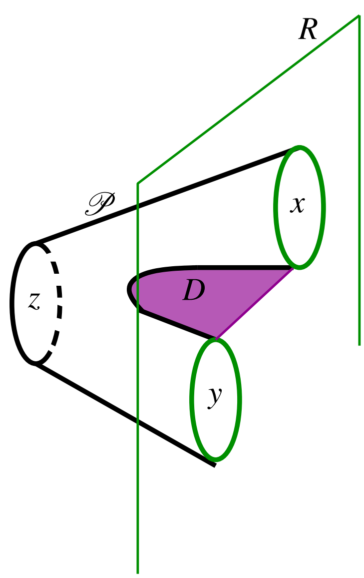

FIGURE 5.6. The disc D in Case 2.2

Case 3: b is non-separating and $q^* = 0$. Since b is non-separating, $\eta(\overline{\beta}) - \overline{Q}$ consists of copies of $D^2 \times I$ labelled $W_1, \ldots, W_{q-1}$ which are separated by discs $\beta_1, \ldots, \beta_q$ each parallel to β so that each W_i is adjacent to β_i and β_{i+1} where the indices run mod q. $\partial_0 N[b]$ is a single torus T_0. See Figure 5.7.

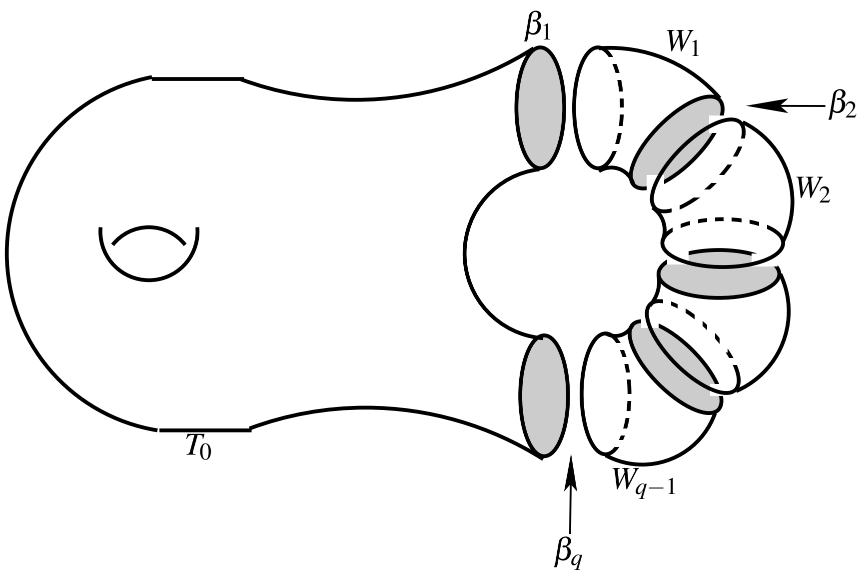

FIGURE 5.7. The solid torus and 1-handles W_j

We need only consider the following cases, as the others are similar to prior cases.

Case 3.4: ε is located on T_0 and either both endpoints are on $\partial\beta_1$ or both are on $\partial\beta_q$. The arc ε is a meridional arc. Suppose, without loss of generality, that $\partial\varepsilon \subset \partial\beta_1$. Boundary-compress $\overline{R} - \mathring{\beta}_1$ along D. This creates a surface $\overline{J}$ with boundary on T_0. After possibly discarding a boundary-parallel annulus $\overline{J}$ is essential and the (C) properties hold as well as (B2) and (B3). We need to show that (B1) can be achieved, if desired.

Suppose that we are in the situation of "refilling meridians" so that $N \subset M$ and F bounds a genus 2 handlebody W in $M - N$ with a and b bounding discs in W. Then since the endpoints of ε are on the same component of $\partial\eta(a) \subset F$, ε is a meridional arc of $b - a$. If $\partial\overline{R}$ is not meridional on T_0 this case, therefore, cannot occur. Thus, the (C) and (B) properties hold.

Case 3.5: ε is located on T_0 and has one endpoint on β_1 and the other on β_q. The disc D guides an isotopy of $\overline{R}$ to a surface $\overline{Q}$ which is suitably embedded in $M[\beta]$ and has $q^*(\overline{Q}) = 1$. We have $\widetilde{q}(\overline{Q}) = \widetilde{q}(\overline{R}) - 1$. The surface $\overline{Q}$ can also be created by boundary-compressing $\overline{R} - (\beta_1 \cup \beta_q)$ with D and then adding a disc β^* to the new boundary component. See Figure 5.6. Clearly, the (C) and (B) properties hold.

The previous cases have each described an operation on $\overline{R}$ which produces an essential surface $\overline{Q}$ having the (C) and (B) properties. Furthermore, the maneuvre described in each case strictly decreases complexity. Thus, after repeating the operation enough times either the surface $\overline{Q}$ will have $\widetilde{q}(\overline{Q}) = 0$ or there will be no a–boundary compressions for Q. That is, the (C) and (B) properties hold and, in addition, (D1) holds.

5.2. Eliminating a–torsion $2g$–gons

We may now assume that there is an a–torsion $2g$–gon D for Q with $g \geq 2$ (since an a–torsion 2–gon is an a–boundary compressing disc). For ease of notation, relabel and let $\overline{R} = \overline{Q}$ and $R = Q$. By the definition of a–torsion $2g$–gon, there is a rectangle E containing the parallel arcs $\partial D \cap F$ which, when attached to R, creates an orientable surface. Two opposite edges of

∂E lie on ∂R and the other two are parallel (as un-oriented arcs) to the arcs of $\partial D \cap F$. Denote the components of ∂R containing the two edges of ∂E by ∂_x and ∂_y. It is entirely possible that $\partial_x = \partial_y$. If ∂_x is a component of $\partial R - \partial \overline{R}$, let β_x denote the disc in $\overline{R} - R$ which it bounds. Similarly define β_y.

Suppose that $\overline{R}$ is a planar surface or 2–sphere. Let $\widehat{N}$ be the 3–manifold obtained from $N[b]$ by attaching 2–handles to $\partial N[b]$ in such a way that each component, but one, of $\partial \overline{J}$ bounds a disc in $\widehat{N}$. Attach these discs to $\overline{R}$ forming a surface $\widehat{R}$. Since $\overline{R}$ was a planar surface or 2–sphere, $\widehat{R}$ is a disc or 2–sphere. A regular neighborhood of $\widehat{R} \cup E$ is a solid torus and the disc D is in the exterior of that solid torus and winds longitudinally around it $n \geq 2$ times. Thus $\eta(\widehat{R} \cup E \cup D)$ is a lens space connected summand of $\widehat{N}$. Hence, redefining $\overline{Q} = \overline{J}$ we satisfy the (C), (B), and (D) properties.

We may, therefore, assume that $\overline{R}$ is not a planar surface or 2–sphere. We need to show that we can achieve (D2) in addition to the (C), (B), and (D1) properties. The surface $\overline{R}' = (\overline{R} - (\beta_x \cup \beta_y)) \cup E$ is compressible by the disc D. Compress it to obtain an orientable surface $\overline{J}$. Notice that

$$(-\chi(\overline{J}), \widetilde{q}(\overline{J})) < (-\chi(\overline{R}), \widetilde{q}(\overline{R})).$$

Analyzing the position of E as we did the position of ε in the previous section and possibly performing the "annulus attachment trick", we can guarantee that the (C) and (B) properties are satisfied. If the ends of E are both on $\partial \overline{R}$ then the boundary of $\overline{J}$ may have different slope from the boundary of $\overline{R}$. Whether or not we perform the annulus attachment trick, the surface

$\overline{J}$ may be inessential. Compressing, boundary compressing, and discarding null-homologous components produces a non-empty essential surface $\overline{Q}$ satisfying properties (B) and (C). Considerations similar to those necessary for achieving (B1) in case 1.4 explain why (B2) is phrased as it is. (B3) is incompatible with (D2) since discarding components may discard $\partial\overline{R} \cap \partial_0 N[b]$ converting a non-empty slope to an empty slope. A future attempt to eliminate an a–boundary compressing disc or a–torsion $2g$–gon may then introduce new boundary components on $\partial_0 N[b]$ of different slope.

As before, complexity has been strictly decreased for both assumptions (I) and (II). Of course, we may now have additional a–boundary compressing discs or a–torsion $2g$–gons to eliminate as in the previous section. Since all these operations lower complexity, the process terminates with the required surface $\overline{Q}$. $\square$

The surface $\overline{Q}$ produced by the previous theorem may be disconnected. (For example, if b is separating it is possible we could start with $\overline{R}$ being a disc with boundary on T_0 and end up with $\overline{Q}$ the union of an annulus with boundary on $T_0 \cup T_1$ and a disc with boundary on T_1.) The next corollary puts our minds at rest by elucidating when we can discard components to arrive at a connected surface $\overline{Q}$.

COROLLARY 5.2. *The following statements are true:*

- *If $\overline{R}$ is a collection of spheres or discs then after discarding components of the surface $\overline{Q}$ created by Theorem 5.1 we may assume that $\overline{Q}$ is an essential sphere or disc such that $\widetilde{q}(\overline{Q}) \leq \widetilde{q}(\overline{R})$ and conclusions (B2), (B3), (D1), (D3), and (D4) hold.*

- *If $N[b]$ does not contain an essential disc or sphere, then we may assume the $\overline{Q}$ produced by Theorem 5.1 to be connected and Conclusions (C1), (C2), (B2), and (D1) - (D4) hold. Furthermore, if $\overline{R}$ is non-separating, so is $\overline{Q}$.*

PROOF. Suppose that $\overline{R}$ is a collection of spheres or a discs and let $\widetilde{Q}$ be the surface produced by Theorem 5.1. Since $-\chi(\overline{R}) < 0$, by conclusions (C1) and (C2) of that theorem, $-\chi(\widetilde{Q}) < 0$ and each component of $\widetilde{Q}$ is a planar surface or $\widetilde{Q}$ is a sphere. Indeed, at least one component $\overline{Q}$ of $\overline{Q}$ is a sphere or disc. By conclusion (D1), either $\widetilde{Q}$ is disjoint from $\overline{\beta}$ or there is no a–boundary compressing disc for $\widetilde{Q} \cap N$. If there is an a–boundary compressing disc for $\overline{Q} \cap N$ then an outermost arc argument shows that there would be one for $\widetilde{Q} \cap N$. Thus, either $\overline{Q}$ is disjoint from $\overline{\beta}$ or there is no a–boundary compressing disc for $\overline{Q}$. As argued in the proof of Theorem 5.1, if there is an a–torsion $2g$–gon for Q, then $N[b]$ contains a lens-space connected summand. It is clear, therefore, that the required conclusions hold.

Suppose that $N[b]$ contains no essential disc or sphere. Let $\widetilde{Q}$ be the surface produced by Theorem 5.1 and notice that $\widetilde{Q}$ contains no disc or sphere components. Choose a component $\widetilde{Q}_0$ of $\widetilde{Q}$ and discard the other components. Neither negative euler charactistic nor $\widetilde{q}$ are raised. If $\overline{R}$ was non-separating, choose $\widetilde{Q}_0$ to be non-separating. Either $\widetilde{Q}_0$ satisfies the conclusion of the Corollary or $\widetilde{q}(\widetilde{Q}_0) > 0$ and there is an a–boundary compressing disc or a–torsion $2g$–gon for $\widetilde{Q}_0 \cap N$. Apply the theorem with $\overline{R} = \widetilde{Q}_0$ and notice that the surface $\widetilde{Q}_1$ produced has strictly smaller complexity. Thus, repeating this process, each time discarding all but one component, we eventually obtain the connected surface $\overline{Q}$ promised by corollary. $\square$

CHAPTER 6

Degenerating Handle Additions

Most of the applications of the main results will concern refilling meridians of genus 2 handlebodies, but first we prove some fairly general results about 2–handle addition to a genus 2 boundary component. These theorems will be proved without using the second sutured manifold theorem. The proofs are very similar, with the second being more difficult.

THEOREM 6.1. *Suppose that F has genus 2, N is compact, orientable, and irreducible, $\partial N - F$ is empty or consists of tori, that N is boundary-irreducible and that there is no essential annulus in N with both boundary components parallel to $a \subset F$ or both boundary components parallel to $b \subset F$. If a and b are separating non-parallel curves, then one of $N[a]$ and $N[b]$ is irreducible.*

PROOF. Suppose that $N[b]$ is reducible. Since there is no essential annulus in N with boundary parallel to a, there is no essential 2-sphere in $N[a]$ minimally intersecting $\overline{\alpha}$ twice. By Lemma 4.1, (N,a) is a taut sutured manifold. Let $\overline{R}$ be an essential sphere or disc in $N[b]$ and apply Theorem 5.1 to obtain a surface $\overline{Q}$. By Corollary 5.2, we may assume that $\overline{Q}$ is a sphere or disc. Since N is irreducible and boundary-irreducible, $\widetilde{q}(\overline{Q}) > 0$. Furthermore, there is no a–boundary compressing disc for Q.

Since b is separating, $q^* = 0$ and since a is separating $\nu = \nu_\partial = 0$. Since a and b are both separating $\Delta \geq 2$. Hence,

$$K(\overline{Q}) = q(\Delta - 2) + \Delta_\partial \geq 0.$$

In particular, $-2\chi(\overline{Q}) < K(\overline{Q})$. There is no essential annulus in N with boundary parallel to a. Hence, $\overline{\alpha}$ does not intersect an essential sphere in $N[a]$ exactly twice without being able to be isotoped to be disjoint from it. By the first sutured manifold theorem, $(N[a], \varnothing)$ must be taut. Therefore, $N[a]$ is irreducible. $\square$

Our second theorem is similar, but has stronger assumptions and conclusions.

THEOREM 6.2. *Suppose that F has genus 2, and that N is simple. Suppose that a and b are non-isotopic separating curves on F. Suppose that $N[a]$ is reducible. Then if $N[b]$ is non-simple, it contains an essential annulus with boundary on non-torus components of $\partial N[b]$ and $\Delta = 4$.*

PROOF. Notice, first, that since a and b are separating, Δ is even and positive. If Δ were equal to two then $a - b$ would have a single arc on each once-punctured torus component of $F - b$, implying that a was non-separating. Thus, $\Delta \geq 4$.

Now suppose that $N[b]$ contains a surface $\overline{R}$ which is an essential sphere, disc, torus, or annulus. Let $\overline{Q}$ be the surface obtained by applying Corollary 5.2 to $\overline{R}$. The surface $\overline{Q}$ is still an essential sphere, disc, annulus, or torus. It is not disjoint from a since N is simple and $\Delta > 0$. Furthermore, there is no

a-boundary compressing disc for Q. We may assume that out of all such $\overline{Q}$, q has been minimized. If $\overline{Q}$ is an annulus, suppose for the time being that $\partial\overline{Q}$ does not have all its boundary components on non-torus components of $\partial N - F$. Since a and b are separating, $\hat{\gamma} = \varnothing$ and $b^* = \varnothing$.

If $\overline{Q}$ has a boundary component on a non-torus component of $\partial N - F$, let c be that component of $\partial\overline{Q}$. Since N is simple, c satisfies the requirements for an application of Lemma 4.1. Let γ be the sutures provided by that lemma.

Notice that $q > 0$ since N is simple. We may now apply the first sutured manifold theorem. Since N does not contain an essential annulus, conclusion (1) does not occur. By the construction of $\overline{Q}$, there is no a–boundary compressing disc for Q. Thus,

$$(\Delta - 2)q + \Delta_\partial \leq -2\chi(\overline{Q}).$$

Hence,

$$\Delta \leq 2 + (-\Delta_\partial + -2\chi(\overline{Q}))/q \leq 2 - \Delta_\partial/q.$$

Since Δ_∂ is non-negative, we have $\Delta = 2$. This contradicts our initial observation that $\Delta \geq 4$.

We may, therefore, assume that $\overline{Q}$ is an annulus with both boundary components on non-torus components of $\partial N - F$. Let G be the components of $\partial N - F$ containing $\partial\overline{Q}$. Let N' be the manifold obtained by doubling N along G. That is, N' is formed by gluing a copy N_2 of N to $N_1 = N$ along G. Let F_i, a_i, b_i, Q_i be the copy of F, a, b, and Q lying in N_i. The gluing should be performed so that $\overline{Q}' = \overline{Q} \cup Q_2$ is a punctured torus in $N'[b_1]$ with punctures on F_2 parallel to b_2. It is easy to show that N' is simple. Notice that $N'[a]$ is reducible.

Let $Q' = \overline{Q'} \cap N'$. Suppose that D is an a_1–boundary compressing disc for Q' with $\varepsilon = \partial D \cap F_1$. Since N_1 and N_2 are simple, we may assume that $D \cap G$ consists of arcs which are essential in Q'. Since there is no a–boundary compressing disc for Q in N, this collection of arcs is non-empty. Since G is disjoint from F_1, there is some arc of $D \cap G$ which is outermost on D and does not contain ε in the outermost disc it bounds. Let E be the outermost disc containing that arc. Then E is a boundary compressing disc for Q_1 or Q_2. Without loss of generality, suppose it to be Q_1. Since $\overline{Q}$ is essential in $N_1[b_1]$, the arc $\partial E \cap \overline{Q}$ must be inessential in $\overline{Q}$. Cutting $\overline{Q}$ along ∂E produces a surface with an annulus component and a disc component. Since $N_1[b_1]$ contains no essential discs, the disc component must be inessential. But this implies that there is an isotopy of $\overline{Q}$ reducing q, contradicting our choice of $\overline{Q}$. Hence, there is no a_1–boundary compressing disc for Q'.

Let $c = b_2$ and apply Lemma 4.1 to construct sutures γ on $\partial N'$ which are disjoint from c so that $(N', \gamma \cup a_1)$ is a taut sutured manifold. Since all boundary components of $\overline{Q}'$ are parallel to b_2, $\Delta_\partial = \nu_\partial = 0$. Also, $-2\chi(\overline{Q}') = 2q$ since $\overline{Q}'$ is a punctured torus with q boundary components. If $-2\chi(\overline{Q}) < K(\overline{Q})$ then the first sutured manifold theorem shows that $N'[a_1]$ is irreducible, a contradiction. Hence, $-2\chi(\overline{Q}) \geq K(\overline{Q})$. Thus,

$$2q \geq q(\Delta - 2).$$

Solving for Δ, we observe $\Delta \leq 4$. Since $\Delta \geq 4$, we conclude $\Delta = 4$. $\square$

REMARK. A separating curve in ∂N is an example of what Scharlemann and Wu [**SW**] call a **basic** curve. They prove that if N is simple and one of a and b is basic, then if $N[a]$ is reducible and $N[b]$ is boundary-reducible then a and b can be isotoped to be disjoint. They conjecture that if both a and b are basic and neither $N[a]$ nor $N[b]$ is simple then $\Delta \leq 5$. Theorem 6.2 gives some evidence for their conjecture.

CHAPTER 7

Refilling Meridians

We now turn to applying the sutured manifold theorems to "refilling meridians". For the remainder, suppose that M is a 3–manifold containing an embedded genus 2 handlebody W. Let $N = M - \mathring{W}$. Let α and β be two essential discs in W isotoped to intersect minimally and non-trivially. Let $a = \partial\alpha$, $b = \partial\beta$, $b^* = \partial\beta^*$, $M[\alpha] = N[a]$, and $M[\beta] = N[b]$. Recall that L_α and L_β are the cores of the solid tori produced by cutting W along α and β respectively. If we need to place sutures $\hat{\gamma}$ on $F = \partial W$ we will do so as described in Section 4. We begin by briefly observing that for any suitably embedded surface $\overline{Q} \subset M[\beta]$, with boundary disjoint from $\gamma \cap \partial M$, $K(\overline{Q}) \geq 0$.

If α is separating,

$$K(\overline{Q}) = q(\Delta - 2) + q^*(\Delta^* - 2) + \Delta_\partial.$$

Since b, b^*, and a all bound discs in W, Δ is at least two. If $q^* \neq 0$, then Δ^* is also at least two. Thus, $K(\overline{Q}) \geq 0$.

Recall (Section 4) that if α is non-separating, any arc of $b - \mathring{\eta}(a)$ with endpoints on the same component of $\partial\eta(a)$ is a meridional arc of $b - a$. The number of these meridional arcs is denoted $\mathscr{M}_a(b)$ and it is always even and always at least two since there are the same number of meridional arcs based at each component of $\partial\eta(a) \subset F$. The sutures $\hat{\gamma}$ are disjoint from these meridional arcs. Since any arc of $b-a$ which is not a meridional arc intersects exactly one suture exactly once, we have

$$\Delta - \nu = \mathscr{M}_a(b) \geq 2$$

and

$$\Delta^* - \nu^* \geq \mathscr{M}_a(b^*) \geq 2.$$

Since $\partial\overline{Q}$ is disjoint from $b \cup b^*$, it is also disjoint from the meridional arcs of $b-a$. Consequently, each arc of $\partial\overline{Q} - a$ intersects $\hat{\gamma}$ at most once. Hence, $\Delta_\partial - \nu_\partial \geq 0$. When α is non-separating, we, therefore, have

$$K(\overline{Q}) \geq q(\mathscr{M}_a(b) - 2) + q^*(\mathscr{M}_a(b^*) - 2) + \Delta_\partial - \nu_\partial \geq 0.$$

Before proceeding to more interesting results, we need to know that there are taut conditioned Seifert surfaces.

LEMMA 7.1. *Suppose that $M = S^3$. Then there is a Seifert surface S for L_α which is disjoint from $\overline{\alpha}$ (i.e. lies in N) and is a taut conditioned surface in N with boundary disjoint from a.*

PROOF. First we show that L_α does contain a conditioned Seifert surface disjoint from $\overline{\alpha}$. Choose a Seifert surface $\Sigma_0 \subset N[a]$ for L_α. If L_α is a link, Σ_0 may not be connected. Since $\partial\Sigma_0$ is a longitudinal on $\partial N[a]$, we may assume (when α is non-separating) that it intersects γ exactly twice.

Calculate the algebraic intersection number between $\overline{\alpha}$ and each component of Σ_0. If it is $n \neq 0$, an endpoint of $\overline{\alpha}$ may be isotoped around $\partial S^3[\alpha]$

creating n intersections of sign $-n/|n|$. Perform the isotopy so that $\partial\overline{\alpha}$ is always disjoint from γ. Rather than isotoping $\overline{\alpha}$, we may instead isotope Σ_0. We take this latter viewpoint. The requirement, from the former viewpoint, that $\partial\overline{\alpha}$ be disjoint from γ guarantees that, from the latter viewpoint, if α is non-separating then $\partial\Sigma_0$ still intersects each component of γ exactly once.

We may, therefore, assume that the intersection number of $\overline{\alpha}$ with each component of Σ_0 is zero. Choosing an arc σ of $\overline{\alpha} - \Sigma_0$ with endpoints creating intersections of opposite sign on the same component of Σ_0, we attach a tube containing σ to Σ_0, decreasing $|\Sigma_0 \cap \overline{\alpha}|$ (but increasing the genus of Σ_0). The algebraic intersection number of $\overline{\alpha}$ and Σ_0 is still zero. Continuing in this manner, we may construct a conditioned Seifert surface Σ for L_α which is disjoint from $\overline{\alpha}$. Out of all Seifert surfaces for L_α which are disjoint from $\overline{\alpha}$ and which have boundary $\partial\Sigma$ choose one of minimal genus and call it S. Then S is a taut conditioned surface in N. □

REMARK. Notice that even though ∂S (where S is the surface created by the previous lemma) is a longitude on $\partial_0 N[a]$ (when α is separating) it may intersect meridional arcs of $b-a$ more than once. It must, however, intersect them at least once. See Figure 7.1 for a depiction of the "spiralling $\partial\overline{\alpha}$" viewpoint.

An easier proof, which is omitted, gives:

LEMMA 7.2. *Suppose that $M = S^3$. Then there is a Seifert surface $S \subset N[a]$ for L_α which is an $\overline{\alpha}$–taut conditioned surface.*

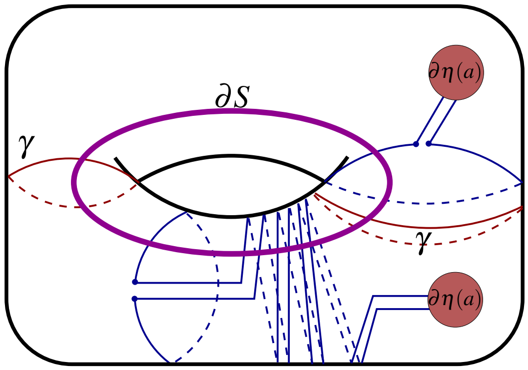

FIGURE 7.1. The result of spiralling $\partial\overline{\alpha}$ around $\partial_0 N[a]$

7.1. Scharlemann's Conjecture

Studying the operation of refilling meridians, Scharlemann [**S5**] was led to the following definitions and conjecture.

Define (M,W) to be **admissible** if

(A0) every sphere in M separates

(A1) M contains no lens space connected summands

(A2) Any two curves in ∂M which compress in M are isotopic in ∂M

(A3) $M - W$ is irreducible

(A4) ∂M is incompressible in N.

He conjectured

CONJECTURE. If (M,W) is admissible then one of the following occurs

- $M = S^3$ and W is unknotted (i.e. N is a handlebody)
- At least one of $M[\alpha]$ and $M[\beta]$ is irreducible and boundary-irreducible
- α and β are "aligned" in W.

The definition of "aligned" is rather complicated and is not needed for what follows, so I will not define it here. I will only remark that it is a notion which is independent of the embedding of W in M.

Scharlemann proved the following:

THEOREM (Scharlemann).

- *If ∂W compresses in N then the conjecture is true.*
- *If $\Delta \leq 4$ then the conjecture is true.*
- *If α is separating and M contains no summand which is a non-trivial rational homology sphere then one of $M[\alpha]$ and $M[\beta]$ is irreducible and boundary-irreducible.*
- *If both α and β are separating then the conjecture is true. If, in addition, $\Delta \geq 6$ one of $M[\alpha]$ and $M[\beta]$ is irreducible and boundary-irreducible.*

With a slight variation on the notion of "admissible", Scharlemann's Conjecture can now be completed for a large class of manifolds.

Define the pair (M,W) to be **licit** if the following hold:

(L0) $H_2(M) = 0$.

(L1) $H_1(M)$ is torsion-free.

(L2) No curve on a non-torus component of ∂M which compresses in M bounds an essential annulus in N with a meridional curve of ∂W (that is, a curve on ∂W which bounds a disc in W).

(L3) N is irreducible

(L4) ∂M is incompressible in N.

The major improvement provided by the next theorem is that the case of non-separating meridians can be effectively dealt with. The theorem completes Scharlemann's conjecture for pairs (M,W) which are both licit and admissible.

THEOREM 7.3 (Modified Scharlemann Conjecture). *Suppose that (M,W) is licit and that α and β are two essential discs in W. Make the following incompressibility assumptions:*

- *If α is separating, then $\partial W - a$ is incompressible in N.*
- *If β is separating, then $\partial W - b$ is incompressible in N.*
- *If α is non-separating, then there is no essential disc in $M[\alpha]$ which is disjoint from $\overline{\alpha}$.*
- *If β is non-separating, then there is no essential disc in $M[\beta]$ which is disjoint from $\overline{\beta}$.*

Then either α and β can be isotoped to be disjoint or all of the following hold:

- *One of $M[\alpha]$ or $M[\beta]$ is irreducible*
- *If one of $M[\alpha]$ or $M[\beta]$ is reducible then no curve on ∂M compresses in the other.*
- *No curve on ∂M compresses in both $M[\alpha]$ and $M[\beta]$.*
- *If $\partial M = \varnothing$ then one of $M[\alpha]$ or $M[\beta]$ is irreducible and boundary-irreducible (i.e. not a solid torus).*

The theorem would certainly be easier to state if we replaced the incompressibility assumptions with the assumption that ∂W was incompressible in N. However, we require the stated assumptions later. Conditions (L0) and (L1) are stronger than Conditions (A0) and (A1) but are used to guarantee that $H_1(M[\alpha])$ and $H_1(M[\beta])$ are torsion-free; this is required for the application of the second sutured manifold theorem. Condition (L2) is neither stronger nor weaker than Condition (A2) since we allow multiple curves on ∂M to compress in M but forbid the existence of certain annuli. To show that some condition like (A2) was required, Scharlemann points out the following example:

EXAMPLE. Let M be a genus 2–handlebody and let $W \subset M$ so that $M - \mathring{W}$ is a collar on ∂W. (That is, M is a regular neighborhood of W.) Then conditions (A0), (A1), (A3), (A4), (L0), (L1), (L3), and (L4) are all satisfied. But given any essential disc $\alpha \subset W$, $M[\alpha]$ is obviously boundary-reducible. Both (A2) and (L2) rule out this example.

PROOF. Suppose, without loss of generality, that $M[\beta]$ is reducible or boundary-reducible. We begin by showing that $H_1(M[\alpha])$ is torsion-free. Consider M as the union of $V = W - \mathring{\eta}(\alpha)$ and $M[\alpha]$. Using assumption (L0) that $H_2(M) = 0$, we see that the Mayer-Vietoris sequence gives the exact sequence:

$$0 \to H_1(\partial V) \xrightarrow{\phi} H_1(M[\alpha]) \oplus H_1(V) \xrightarrow{\psi} H_1(M) \to 0.$$

Suppose that x is an element of $H_1(M[\alpha])$ and that $n \in \mathbb{N}$ is such that $nx = 0$. Then $n\psi(x,0) = \psi(nx,0) = 0$. Since $H_1(M)$ is torsion-free, $\psi(x,0) = 0$. Thus, by exactness, $(x,0)$ is in the image of ϕ. Let $y \in H_1(\partial V)$ be in the preimage of $(x,0)$. Also, $\phi(ny) = n\phi(y) = (nx,0) = (0,0)$. From exactness, we know that ϕ is injective. Hence, $ny = 0 \in H_1(\partial V)$. The boundary of V is a collection of tori and, therefore, $H_1(\partial V)$ is torsion-free. Consequently, $y = 0$. Therefore, $x = 0$ and $H_1(M[\alpha])$ is torsion-free.

Assume that $\Delta > 0$. We will now show that $M[\alpha]$ is irreducible and that if a curve on ∂M compresses in $M[\beta]$ then it does not compress in $M[\alpha]$ and that if $M[\beta]$ is reducible then no curve of ∂M compresses in $M[\alpha]$. If ∂M is compressible in $M[\beta]$, let c_β be a curve on ∂M which compresses in $M[\beta]$. If $c_\beta = \varnothing$, let c be any curve on ∂M which compresses in M, otherwise let $c = c_\beta$.

By Lemma 4.1 and our incompressibility assumptions, we may choose sutures γ on $\partial M[\alpha]$ so that $\hat{\gamma} = \gamma \cap \partial_0 M[\alpha]$ is chosen as usual and so that $\gamma \cap c = \varnothing$ and $(M[\alpha], \gamma)$ is an $\overline{\alpha}$–taut sutured manifold. Let $\overline{R}$ be either an essential sphere, an essential disc with boundary $c_\beta = c$, or an essential disc with boundary on $\partial_0 M[\beta]$. Let $\overline{Q}$ be the result of applying Corollary 5.2 to $\overline{R}$. $\overline{Q}$ is an essential sphere, an essential disc with boundary c_β, or an essential disc with boundary on $\partial_0 M[\beta]$. By the irreducibility of N and the incompressibility assumptions, $\widetilde{q}(\overline{Q}) > 0$. Consequently, by Corollary 5.2, there are no a–boundary compressing discs or a–torsion $2g$–gons. Since $K(\overline{Q}) \geq 0$ and $-2\chi(\overline{Q}) < 0$, by the second sutured manifold theorem $(M[\alpha], \gamma)$ is $\varnothing$–taut. In particular, $M[\alpha]$ is irreducible and c does not compress in $M[\alpha]$ since otherwise $R_\pm(\gamma)$ would not be taut in $M[\alpha]$.

Thus, either the theorem is true or both $M[\alpha]$ and $M[\beta]$ are irreducible but have boundary-compressing discs on ∂W. Suppose the latter. Since $\partial_0 M[\alpha]$ and $\partial_0 M[\beta]$ are both collections of tori, the presence of a compressing disc implies either reducibility or that $M[\alpha]$ and $M[\beta]$ are solid tori. Thus, we may assume both are solid tori. This implies that $M = S^3$. By the first sutured manifold theorem and Lemma 7.1, there is a taut conditioned Seifert surface for L_α which is disjoint from $\overline{\alpha}$. Since $M[\alpha]$ is a solid torus, this surface must be a disc lying in N. This, however, contradicts the incompressibility assumptions. Thus, $M[\alpha]$ is not a solid torus, and the theorem is true. $\square$

REMARK. At the cost of adding hypotheses on the embedding of W in M, the conditions for being "licit" can be significantly weakened. For example, the hypotheses on the curves c, a, and b of Lemma 4.1 can be substituted for (L2). An examination of the homology argument at the beginning of the proof shows that (L0) can be be replaced with the assumption that L_α and L_β are null-homologous in M. Another way of changing assumptions would be to make greater use of the first sutured manifold theorem which does not require that $M[\alpha]$ be torsion-free in first homology. The next theorem provides an example.

THEOREM 7.4. *Suppose that any two curves of ∂M which compress in M are on the same component of ∂M. Suppose that W is a genus two handlebody embedded in M such that W intersects every essential sphere in M at least three times and every essential disc at least two times. Suppose also that $N = M - \mathring{W}$ is irreducible. Let α and β be essential discs in W which cannot be isotoped to be disjoint. Assume that $M[\alpha]$ and $M[\beta]$ contain no essential disc which is contained in N and that $\partial\alpha$ and $\partial\beta$ do not compress in N. Then the following hold:*

- *One of $M[\alpha]$ and $M[\beta]$ is irreducible and is not a solid torus*
- *If one of them is reducible the other is boundary-irreducible.*
- *If $c_a \subset \partial M$ is a curve which compresses in $M[\alpha]$ and if $c_b \subset \partial M$ is a curve which compresses in $M[\beta]$ then c_a and c_b cannot be isotoped in ∂M to be disjoint.*

PROOF. Without loss of generality, assume that $M[\beta]$ is reducible or boundary-reducible and let $\overline{Q}$ be an essential sphere or disc obtained by applying Corollary 5.2, as before. If $\partial\overline{Q}$ is on ∂M then we may assume that $\partial\overline{Q} = c_b$. Let $T = T(\gamma)$ be the torus components of ∂M.

We need to place sutures on ∂M. To do this, we'll define curves c that can be used in Lemma 4.1. If $\overline{Q}$ is a sphere or disc with boundary on $\partial_0 M[\beta] \cup T$, define $c_\beta = \varnothing$. Otherwise, let $c_\beta = c_b$. If no curve of ∂M disjoint from c_b compresses in $M[\alpha]$, then let $c_\alpha = \varnothing$. If c_b compresses in $M[\alpha]$, let $c_\alpha = c_b$. If c_b does not compress in $M[\alpha]$ but a curve c_a disjoint from c_b does, let $c_\alpha = c_a$. Define $c = c_\alpha \cup c_\beta$ and notice that if $|c| = 2$, there is no essential annulus in N with boundary equal to c. Also, if a component of c bounds an essential annulus with a curve of $\hat{\gamma} \cup a$ then, because, the components of $\hat{\gamma} \cup a$ bound discs in W, W would intersect a compressing disc for ∂M exactly once. This is forbidden by our hypotheses. Furthermore, if $|c| = 2$ then one component, c_α, of c bounds a disc in $M[\alpha]$ and the other component $c_\beta = c_b$ does not. If a is separating, $|c| = 2$, and $c \cup a$ bounds an essential thrice-punctured sphere in $M[\alpha]$, then attaching discs to c_α and to a shows that $c_\beta = c_b = c - c_\alpha$ compresses in $M[\alpha]$, but this contradicts the construction of c_α. Thus c satisfies the criteria for an application of Lemma 4.1. Let $\gamma = \widetilde{\gamma} \cup \hat{\gamma}$ be the sutures on $\partial M[\alpha]$ provided by that Lemma.

If $M[\alpha]$ is reducible or if $c_\alpha \neq \varnothing$ then $(M[\alpha], \gamma)$ is not taut. If $M[\alpha]$ is a solid torus, then, by our hypotheses, every taut conditioned surface with boundary on $\partial_0 M[\alpha]$, of which there is one (Lemma 7.1), intersects $\overline{\alpha}$. We can, therefore, apply the first sutured manifold theorem. Since $\overline{Q}$ is a disc or sphere, $-2\chi(\overline{Q}) < K(\overline{Q})$. By the construction of $\overline{Q}$, there is no a–boundary compressing disc for Q in $N = M - \mathring{W}$. Thus, $M[\alpha]$ contains an essential separating sphere S intersecting $\overline{\alpha}$ twice and which cannot be isotoped to intersect $\overline{\alpha}$ fewer times. The sphere S bounds a non-trivial homology ball.

Because W intersects every essential sphere in M at least three times, S cannot be an essential sphere for M. Let B be the ball in M which S bounds. Notice that this implies that M is a non-trivial homology sphere. Since S is separating and B is not contained in $M[\alpha]$, $\partial_0 M[\alpha] \subset B$. Attaching $\eta(\overline{\alpha})$ to B produces a solid torus V containing W, with ∂V compressible in $V[\alpha]$ and $V - \mathring{W}$ irreducible. Notice that (V, W) is licit. Thus, we may apply the Modified Scharlemann Conjecture to conclude that $V[\beta]$ is irreducible and that $\partial V[\beta]$ does not compress in $V[\beta]$. Thus, $\overline{Q}$ intersects ∂V and an innermost disc of intersection D on $\overline{Q}$ is a compressing disc for ∂V contained outside V. (Inessential curves of intersection should first be eliminated by an innermost disc argument.) If ∂D intersected a meridian curve on ∂V exactly once, ∂D would run exactly once along a regular neighborhood of $\overline{\alpha}$. D then guides an isotopy of $\overline{\alpha}$ into B, contradicting the construction of S. If ∂D is a meridional curve of ∂V, then W is contained in an $S^1 \times S^2$ summand of M. If ∂D intersects every meridional curve of ∂V more than once then

W is contained in a lens space connected summand of M. By hypothesis, W intersects every reducing sphere in M, so M is $S^1 \times S^2$ or a lens space. Both possibilities contradict our previous conclusion that M was a homology sphere. Hence, $M[\alpha]$ is irreducible, $\partial_0 M[\alpha] \cup T$ is incompressible in $M[\alpha]$, and $c_\alpha = \varnothing$. $\square$

The next section contains more applications of the first sutured manifold theorem.

7.2. Essential surfaces in the exteriors of bored unknots and split links

Recall that if α is an essential disc in W which cannot be isotoped to be disjoint from β then L_β is obtained from L_α by boring (and *vice versa*). Our first result generalizes a property of tunnel number 1 knots.

THEOREM 7.5. *Suppose that L_α is a knot or link in S^3 obtained by boring a knot or link L_β using handlebody W. Suppose that either α is non-separating or that $\partial W - \partial\alpha$ is incompressible in N. Suppose also that one of the following holds:*

- *L_β is an unknot*
- *L_β is a split link and $\partial W - \partial\beta$ is incompressible in N.*

Then there is a minimal genus Seifert surface for L_α which is disjoint from $\overline{\alpha}$.

PROOF. If α is non-separating, $\gamma \neq \varnothing$. Since $\partial W - (a \cup \gamma)$ consists of two thrice-punctured spheres each with meridional boundary, it is incompressible in N. If α is non-separating, by hypothesis $\partial W - \partial\alpha$ is incompressible in N. Thus, in either case, by Lemma 4.1, $(N, \gamma \cup a)$ is taut. Let $\overline{R}$ be an essential disc or sphere in $S^3[\beta]$ and let $\overline{Q}$ be the disc or sphere provided by Corollary 5.2. If L_β is a split link then since $\partial W - \partial\beta$ is incompressible in N, $\widetilde{q} > 0$. If $\widetilde{q} = 0$ then L_β is an unknot and $\overline{Q}$ is disjoint from $\overline{\beta}$, but since it is a disc, there is no a–boundary compressing disc for it. Furthermore, in this case, $\partial\overline{Q}$ must intersect the meridional arcs of $a - b$. Thus, whether or not $\widetilde{q}$ is zero, Q has no a–boundary compressing discs and is not disjoint from a. Recall that $-2\chi(\overline{Q}) < 0 \leq K(\overline{Q})$.

By the first sutured manifold theorem and Lemma 7.1, L_α has a minimal genus Seifert surface disjoint from $\overline{\alpha}$ (that is, contained in N). □

COROLLARY 7.6 ([**ST2**, Proposition 4.2]). *If $\overline{\alpha}$ is a tunnel for a tunnel number one knot or link L_α, L_α has a minimal genus Seifert surface disjoint from $\overline{\alpha}$.*

PROOF. As noted in the introduction, every tunnel number one knot or link can be obtained by boring an unknot L_β using the standard unknotted genus two handlebody in S^3. Conversely, a tunnel for a non-trivial tunnel number one knot or link is a boring arc for converting the knot or link into the unknot L_β. Thus, unless α is separating and $\partial W - \partial\alpha$ is compressible in N, the corollary follows immediately from Theorem 7.5.

We may, therefore, assume that L_α is a split link. The surface ∂W is a genus two Heegaard surface for $S^3[\alpha]$. If L_α is a split link, $S^3[\alpha]$ contains an essential sphere, so by Haken's Lemma for Heegaard splittings there is an essential sphere P intersecting the Heegaard surface in a single loop. One side of the Heegaard surface is a compressionbody with two boundary components, each a torus. Thus, P must intersect that compressionbody in the unique (up to isotopy) essential disc. That disc is parallel to α. $\partial W - \mathring{\eta}(P)$ has two components each of which is a genus one Heegaard splitting for the exterior of a component of L_α. The only knot with a genus one Heegaard splitting for its exterior is the unknot and so L_α is the unlink of two components. Since the connected sum of Heegaard splittings is well-defined $\overline{\alpha} \cap (S^3 - \mathring{\eta}(P))$ consists of two unknotted arcs. Thus, each component of L_α bounds a disc disjoint from $\overline{\alpha}$ and the corollary is proved when L_α is a split link. $\square$

REMARK. The proof of the previous corollary is not any better than Scharlemann and Thompson's proof. Indeed, their proof is certainly easier to understand than the arguments of this paper. However, it is interesting to note that they do rely on a theorem of Gabai which was proved using sutured manifold theory. The point of Theorem 7.5 is that a rather significant property of tunnel number one knots has a natural generalization to knots and links obtained by boring an unknot.

Using Theorem 7.5, we can reverse the roles of α and β to obtain:

THEOREM 7.7. *Suppose that $L_\beta \subset S^3$ is obtained by boring a split link or unknot L_α. If L_α is a split link, assume that $\partial W - \partial \alpha$ is incompressible* *in N. If L_α is an unknot, assume that there does not exist an essential disc in $S^3[\alpha]$ disjoint from $\overline{\alpha}$. Then L_β is not a split link or unknot and L_β has a minimal genus Seifert surface $\overline{Q}$ properly embedded in $S^3[\beta]$, which is disjoint from $\overline{\beta}$ and for which one of the following is true:*

- $-2\chi(\overline{Q}) \geq \Delta_\partial - \nu_\partial$
- *There is an a–boundary compressing disc for $\overline{Q}$ in N*

REMARK. Corollary 8.4 rephrases this theorem for rational tangle replacements. Following that theorem, there is an example which shows that the possibility that there is an a–boundary compressing disc for $\overline{Q}$ cannot be eliminated. Notice that if $\overline{\beta}$ is isotopic with fixed endpoints to a non-trivial arc in $\overline{Q}$ then there is an a–boundary compressing disc for $\overline{Q}$ in S^3.

PROOF. The Modified Scharlemann Conjecture shows that L_β is not a split link or unknot.

By Theorem 7.5, applied with α and β reversed, there is a minimal genus Seifert surface $\overline{Q}$ for L_β which is disjoint from $\overline{\beta}$; that is, it is contained in N. The only way in which $\overline{Q}$ could be disjoint from the meridional arcs of $a - b$ is if β were separating and $\overline{Q}$ had boundary on a single component of L_β. This contradicts the definition of Seifert surface for L_β, so $\overline{Q}$ is not disjoint from $\eta(a)$.

If there is an a–boundary compressing disc for $\overline{Q}$ in N, we are done, so suppose that no such disc exists. If $-2\chi(\overline{Q}) < K(\overline{Q})$ the first sutured manifold theorem and Lemma 7.1 imply that $S^3[\alpha]$ is irreducible and that there is a minimal genus Seifert surface for L_α which is disjoint from $\overline{\alpha}$. The first option means that L_α isn't a split link and the second that L_α isn't an unknot since $\partial W - \partial\alpha$ is incompressible. Hence, $-2\chi(\overline{Q}) \geq K(\overline{Q})$. Since $\overline{Q}$ is disjoint from $\overline{\beta}$, $q = q^* = 0$. The given inequality follows from the definition of $K(\overline{Q})$. $\square$

With the stronger assumption that ∂W is incompressible in N, we can restrict the possibilities for obtaining a non-hyperbolic knot or link from a split link by boring.

THEOREM 7.8. *Suppose that L_β is a knot or link obtained by boring the link L_α using a handlebody $W \subset S^3$ with $N = S^3 - \mathring{W}$ boundary-irreducible. Suppose that L_α is a split link or that there is no minimal genus Seifert surface for L_α disjoint from $\overline{\alpha}$. If the exterior of L_β contains an essential annulus or torus then one of the following holds:*

(1) There is an essential torus in N

(2) There is an essential annulus in the exterior of L_β disjoint from $\overline{\beta}$ and which is either disjoint from or has meridional boundary on some component of L_β.

(3) $\Delta = 2$ and if there is an essential annulus then there is one which is either disjoint from or has meridional boundary on some component of L_β.

EXAMPLE. Figure 1.6 shows that a composite knot can be obtained from a split link by a band sum. Thickening the band and the unknot gives us W, and the exterior of W is boundary-irreducible. This shows that the third case can arise.

Both versions of the second conclusion are possible. Figure 7.2 shows a spine for a genus two handlebody. The "S"-shaped arc is disjoint from an essential meridional annulus A. Refilling the meridian of that arc creates a split link with one component a trefoil and the other component an unknot. It is not hard to show that the exterior of the handlebody is boundary-irreducible. Using the "S" shaped arc to perform a band-sum creates a knot L_β which is the connected sum of a trefoil and a 6_1 knot[1].

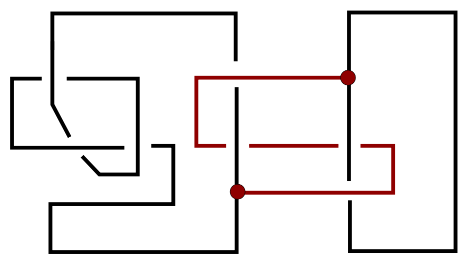

FIGURE 7.2. Performing a rational tangle replacement on the "S" shaped arc leaves the meridional annulus untouched.

Figure 7.3 shows a split link L_α consisting of a trefoil (drawn so the "cabling" annulus is visible) and an unknot. There is an "S" shaped arc joining them. On the trefoil the annulus has boundary slope ± 6. Use the "S"-shaped arc to perform a Kirby band move of the unknot over the trefoil (giving the trefoil a framing of ± 6). We now have a new link L_β with one component the trefoil. By construction the cabling annulus for the trefoil persists into L_β. It is not difficult to show that the exterior of the handlebody is boundary-irreducible.

It is easy to use a "satellite construction" to concoct an example of the first possibility. Figure 7.4 shows a spine for a genus two handlebody W inside

[1]Thanks to Jiho Kim's KnotSketcher and Charles Livingston's KnotFinder for help with this calculation.

a knotted solid torus ∂V. Cutting the edge of the spine containing the local trefoil produces the unlink L_α in S^3. By the Modified Scharlemann Conjecture, ∂V remains essential in the exterior of any knot or link L_β obtained from L_α by boring using W. It is easy to show that ∂W is incompressible in both $V - \mathring{W}$ and $S^3 - \mathring{W}$.

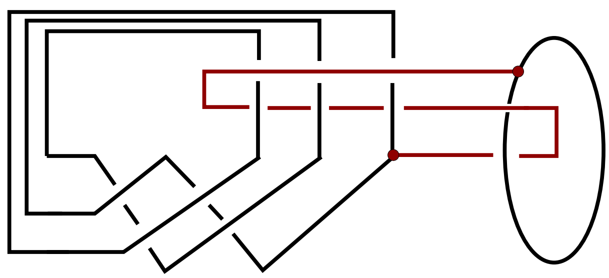

FIGURE 7.3. Performing a Kirby band move using the "S" shaped arc leaves the trefoil's essential annulus untouched.

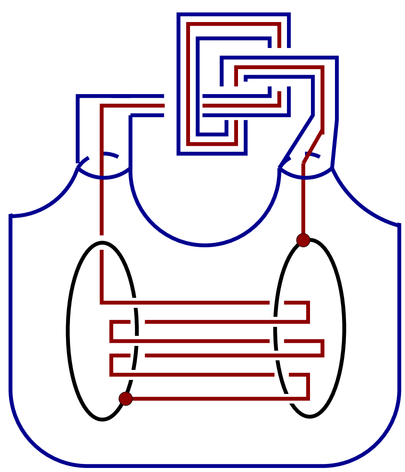

FIGURE 7.4. An essential torus in the exterior of W.

PROOF OF THEOREM 7.8. Suppose that there is no essential torus in N. The Modified Scharlemann Conjecture shows that L_β is not an unknot or split link; consequently, there is no essential disc or sphere in $S^3[\beta]$.

Let $\overline{R}$ be an essential annulus or torus in $S^3[\beta]$ and apply Corollary 5.2, obtaining a connected surface $\overline{Q}$. Since $\overline{Q}$ is not a sphere or disc and since $-\chi(\overline{Q}) \leq -\chi(\overline{R})$, $\overline{Q}$ is an annulus or torus. Since the genus of $\overline{Q}$ is no higher than the genus of $\overline{R}$, if $\overline{R}$ was an annulus, then $\overline{Q}$ is an annulus. If $\overline{Q}$ is disjoint from $\overline{\beta}$ then it is contained in N and must be an annulus by our initial assumption that N contains no essential torus. In this case, if there is an a–boundary compression for $\overline{Q}$, N would contain an essential disc, contradicting the assumption that ∂W is incompressible in N.

We may, therefore, assume that there is no a–boundary compressing disc for $\overline{Q}$. If $\overline{Q}$ is completely disjoint from a, then $\overline{Q}$ is an annulus which is disjoint from the meridional arcs of $a - b$. From our observations about meridional arcs, this means that $\overline{Q} \subset N$ is an annulus which is either disjoint from or has meridional boundary on one component of $\partial S^3[\beta]$.

Suppose, therefore, that $\overline{Q}$ is not completely disjoint from a. Notice that because α is separating, $\overline{\alpha}$ must intersect any reducing sphere for $S^3[\alpha]$ an odd number of times. Thus, by the first sutured manifold theorem, $-2\chi(\overline{Q}) \geq K(\overline{Q})$. Since $\chi(\overline{Q}) = 0$ and since $K(\overline{Q}) \geq 0$ we have $K(\overline{Q}) = 0$. That is,

$$q(\Delta - 2) + q^*(\Delta^* - 2) + \Delta_\partial = 0.$$

Since each term is non-negative, each term must be zero. Hence $\Delta_\partial = 0$, implying that either $\overline{Q}$ is a torus or it is an annulus with boundary disjoint from or consisting of meridians on some component of $\partial S^3[\beta]$. If $q^* \neq 0$, then β is non-separating and we must have $\Delta^* = 2$. Since b^* intersects each meridional arc of $a - b$ at least twice, this means that there is exactly one such meridional arc. The number of meridional arcs is even, so this is a contradiction. If $q \neq 0$ then we have $\Delta = 2$. If both q and q^* are equal to zero, then since $\Delta_\partial = 0$, $\overline{Q}$ is an annulus disjoint from a, a possibility we have already considered. $\square$

In the next section, we study rational tangle replacement as a particular type of boring.

CHAPTER 8

Rational Tangle Replacement

Suppose that L_β is a knot or link in S^3 and that $B' \subset S^3$ is a ball intersecting L_β in two strands r_β so that (B', r_β) is a rational tangle. We will always assume that no component of L_β is disjoint from B'. If (B', r_α) is any other rational tangle, then the knot or link $L_\alpha = (L_\beta - B') \cup r_\alpha$ is obtained by a **rational tangle replacement** on L_β. Let $(B, \tau) = (S^3 - \mathring{B}', L_\beta - \mathring{B}')$ be the complementary tangle. In section 1.4, the terminology associated to rational tangle replacement was defined. We now briefly recall some of this terminology and notation.

Let α and β be trivializing discs for r_α and r_β respectively (isotoped to intersect minimally) and let $W = \eta(L_\beta) \cup B' = \eta(L_\alpha) \cup B'$. Notice that if α and β are not disjoint then L_β and L_α are related by boring using boring handlebody W. The **distance** between r_α and r_β is defined to be $d = \Delta/2$. Since S^3 is prime and, therefore, has no non-trivial homology sphere connected summands, the first sutured manifold theorem is particularly useful. Let $N = S^3 - \mathring{W} = B - \mathring{\eta}(\tau)$.

Before stating the applications, we state and prove some lemmas which allow the terminology of tangle sums and rational tangle replacement to be converted into the terminology of boring.

8.1. Boring and Rational Tangle Replacement

LEMMA 8.1. *Let (B, τ) be a tangle. Suppose that c is an essential separating curve on $\partial B - \tau$. If $\partial N - c$ is compressible in N then c compresses in N.*

PROOF. Let d be an essential curve in $\partial N - c$ which bounds a disc $D \subset N$. Since c is separating and ∂N has genus two, d is a curve in a once-punctured torus. Thus, it is either non-separating or parallel to c. In the latter case, we are done, so suppose that d is non-separating. Let D_+ and D_- be parallel copies of D so that d is contained in an annulus between ∂D_+ and ∂D_-. Use a loop which intersects d exactly once to band together D_+ and D_-, forming a disc D'. The boundary of D' is an essential separating curve in the once-punctured torus. $\partial D'$ is, therefore, parallel to c. Hence, c compresses in N. □

LEMMA 8.2. *Suppose that (B, τ) and (B', r_α) are tangles embedded in S^3 with (B', r_α) a rational tangle so that $\partial B = \partial B'$ and $\partial \tau = \partial r_\alpha$. Suppose that (B', r_β) is rational tangle of distance at least one from (B', r_α). Define the sutures $\gamma \cup a$ on ∂N as before. If*

- *α is non-separating in the handlebody $W = B' \cup \eta(\tau)$, or*
- *if (B, τ) is a prime tangle, or*
- *if (B, τ) is a rational tangle and $\partial \alpha$ does not bound a trivializing disc for (B, τ), or*
- *if $\partial \alpha$ does not compress in (B, τ)*

then $\partial W - (\gamma \cup a)$ *is incompressible in* N. *Consequently,* $(N, \gamma \cup a)$ *is* $\varnothing$*–taut and* $(N[a], \gamma)$ *is* $\overline{\alpha}$*–taut.*

PROOF. If α is non-separating then any compressing disc for $\partial W - (\gamma \cup \partial \alpha)$ would have meridional boundary, implying that S^3 had a non-separating 2–sphere. Thus, we may suppose that α is separating. If (B, τ) is prime, there is no disc separating the strands of τ. Similarly, if (B, τ) is a rational tangle but a does not bound a trivializing disc then a does not compress in (B, τ). Thus, for the remaining three hypotheses, we may assume that a does not compress in (B, τ). By Lemma 8.1, $\partial N - a$ is incompressible in N, as desired. By Lemma 4.1, $(N, \gamma \cup a)$ is taut and $(N[a], \gamma)$ is $\overline{\alpha}$–taut. □

One pleasant aspect of working with rational tangle replacements is that we can make explicit calculations of $K(\overline{Q})$. Here are two lemmas which we jointly call the **Tangle Calculations**.

TANGLE CALCULATIONS I (β separating). *Suppose that* L_β *is a link obtained from* L_α *by a rational tangle replacement of distance* d *using* W. *Let* $\overline{Q}$ *be a suitably embedded surface in the exterior* $S^3[\beta]$ *of* L_β. *Let* $\partial_1 \overline{Q}$ *be the components of* $\partial \overline{Q}$ *on one component of* $\partial S^3[\beta]$ *and* $\partial_2 \overline{Q}$ *be the components on the other. Let* n_i *be the minimum number of times a component of* $\partial_i \overline{Q}$ *intersects a meridian of* $\partial S^3[\beta]$.

- *If* L_α *is a link then*

$$K(\overline{Q}) \geq 2q(d-1) + d(|\partial_1 \overline{Q}| n_1 + |\partial_2 \overline{Q}| n_2).$$

- *If L_α is a knot then*

$$K(\overline{Q}) \geq 2q(d-1) + (d-1)(|\partial_1 \overline{Q}|n_1 + |\partial_2 \overline{Q}|n_2).$$

PROOF. Since L_β is a link, β is separating. Thus, $q^* = 0$. Since a and b are contained in $\partial B' = \partial B$ every arc of $b - a$ is an meridional arc. Hence, $\nu = 0$. By definition $2d = \Delta$.

Let T be a component of $\partial S^3[\beta]$. Without loss of generality, suppose that the components of $\partial \overline{Q}$ on T are $\partial_1 \overline{Q}$. Since every arc of $a - b$ is meridional, there exist d meridional arcs on each component of $\partial S^3[\beta]$. Thus, each component of $\partial_1 \overline{Q}$ intersects a at least dn_1 times. Each component of $\partial_2 \overline{Q}$ intersects a at least dn_2 times. Consequently, $|\partial_1 \overline{Q} \cap a| \geq |\partial_1 \overline{Q}|n_1 d$. Similarly, $|\partial_2 \overline{Q} \cap a| \geq |\partial_2 \overline{Q}|n_2 d$. Hence,

$$\Delta_\partial \geq d(|\partial_1 \overline{Q}|n_1 + |\partial_2 \overline{Q}|n_2).$$

If α is non-separating, the curves γ are also meridian curves of L_β. Thus, γ is intersected n_i times by each component of $\partial_i \overline{Q}$. Hence, if L_α is a knot,

$$\nu_\partial = |\partial_1 \overline{Q}|n_1 + |\partial_2 \overline{Q}|n_2.$$

The result follows. $\square$

TANGLE CALCULATIONS II (β non-separating). *Suppose that L_β is a knot obtained from L_α by a rational tangle replacement of distance d using W. Let $\overline{Q}$ be a suitably embedded surface in the exterior $S^3[\beta]$ of L_β. Suppose that each component of $\partial \overline{Q}$ intersects a meridian of $\partial S^3[\beta]$ n times.*

- *If L_α is a link then*

$$K(\overline{Q}) \geq 2q(d-1) + 2q^*(2d-1) + 2d|\partial\overline{Q}|n.$$

- *If L_β is a knot then*

$$K(\overline{Q}) \geq 2(d-1)(q+2q^*) + 2(d-1)|\partial\overline{Q}|n.$$

PROOF. These calculations are similar to the calculations of the previous lemma, so we make only a few remarks. First, since b^* and $\partial\eta(b)$ cobound a thrice-punctured sphere, every meridional arc of $a-b$ intersects b^* at least twice. Since every arc of $a-b$ is meridional, there are Δ such arcs. Hence $\Delta^* \geq 4d$. Secondly, if L_α is a knot, then b^* intersects γ twice and b intersects γ not at all. Thus,

$$q(\Delta - \nu - 2) + q^*(\Delta^* - \nu^* - 2) \geq q(2d-2) + q^*(4d-4).$$

The given inequality follows. $\square$

Our last observation concerns the implications of an a–boundary compressing disc.

LEMMA 8.3. *Suppose that $\overline{Q}$ is an incompressible and boundary-incompressible surface in $S^3[\beta]$ disjoint from $\overline{\beta}$. If all components of $\partial\overline{Q}$ are meridians then there does not exist an a–boundary compressing disc joining two components of $\partial\overline{Q}$. If $\partial\overline{Q}$ has components on all components of $\partial S^3[\beta]$ and no component is a meridian, then if there is an a–boundary compression for $\overline{Q}$ in $S^3 - \mathring{W}$ the arc $\overline{\beta}$ is properly isotopic into $\overline{Q}$.*

PROOF. Notice, first, that if all components of $\partial\overline{Q}$ are meridians then $\partial\overline{Q} \cap a = \varnothing$, since all arcs of $a - b$ are meridional. Thus, if all components of $\partial\overline{Q}$ are meridional there can be no a–boundary compressing disc for $\overline{Q}$. Suppose therefore that $\partial\overline{Q}$ intersects each component of $\partial S^3[\beta]$ and that no component of $\partial\overline{Q}$ is a meridian. Let D be an a–boundary compression. Let $\varepsilon = \partial D \cap \partial W$. It is a component of $a - \partial\overline{Q}$. Since $\overline{Q}$ is boundary incompressible in $S^3[\beta]$, the arc runs at least once across $\eta(b)$. Since no component of $\partial\overline{Q}$ is a meridian and since it intersects each component of $\partial S^3[\beta]$, each arc of $a - \partial\overline{Q}$ which runs across $\eta(b)$ does so exactly once. Hence, after pushing ε into W slightly, $\eta(\overline{\beta})$ can be viewed as a regular neighborhood of ε. Then D guides an isotopy of $\overline{\beta}$ into $\overline{Q}$. See Figure 8.1. $\square$

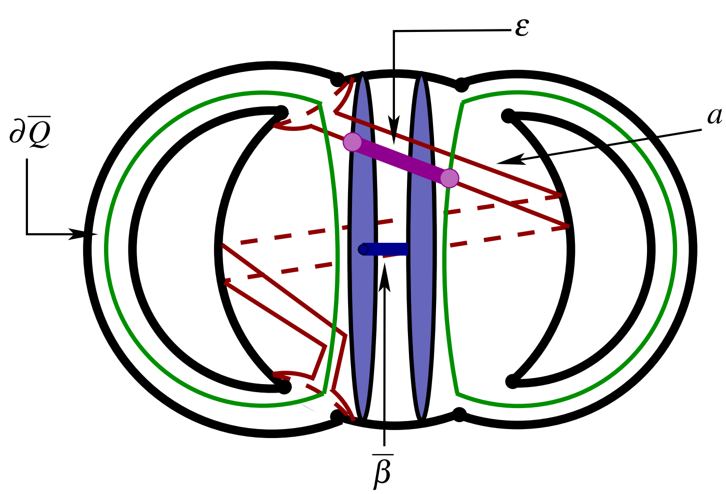

FIGURE 8.1. The arc $\overline{\beta}$ is parallel to ε.

8.2. Seifert surfaces

We begin by restating Theorem 7.7 for rational tangles:

COROLLARY 8.4. *Suppose that $L_\beta \subset S^3$ is obtained by a rational tangle replacement of distance $d \geq 1$ on a split link or unknot L_α. If L_α is a split link, assume that a is incompressible in $B - \tau$. If L_α is an unknot, assume that there does not exist an essential disc in $S^3[\alpha]$ disjoint from $\overline{\alpha}$. Then L_β has a minimal genus Seifert surface $\overline{Q}$ disjoint from $\overline{\beta}$ such that one of the following holds:*

- *$\overline{\beta}$ is properly isotopic into $\overline{Q}$*
- *$-\chi(\overline{Q}) \geq d$ and L_α is a split link*
- *$-\chi(\overline{Q}) \geq d-1$ and L_α is an unknot.*

PROOF. The assumption that if α is separating then a is incompressible in $B - \tau$ implies (Lemma 8.2) that $\partial N - a$ is incompressible in N. Applying Theorem 7.7, we produce the Seifert surface $\overline{Q}$ and either there is an a–boundary compressing disc for $\overline{Q} \subset N$ or $-2\chi(\overline{Q}) \geq K(\overline{Q})$. If the former happens, by Lemma 8.3, we conclude that $\overline{\beta}$ is properly isotopic into $\overline{Q}$. Suppose, therefore, that $-2\chi(\overline{Q}) \geq K(\overline{Q})$. Using the Tangle Calculations and the fact that $q = q^* = 0$ we see that if L_α is a link, then $-2\chi(\overline{Q}) \geq 2d$. If L_α is a knot, then $-2\chi(\overline{Q}) \geq 2(d-1)$. The given inequalities follow immediately. $\square$

A pleasing corollary is Gabai and Scharlemann's result that genus is superadditive under band sum. A band sum is a rational tangle replacement of distance 1 on a split link.

COROLLARY 8.5 (Gabai [**G2**], Scharlemann [**S3**]). *Suppose that* $K_1 \#_b K_2$ *is the band sum of knots* K_1 *and* K_2*. Then*

$$\text{genus}(K_1 \#_b K_2) \geq \text{genus}(K_1) + \text{genus}(K_2)$$

with equality only if K_1 *and* K_2 *have minimal genus Seifert surfaces disjoint from the band.*

PROOF. The statement holds if the band sum is a connected sum (i.e. if the band intersects a splitting sphere exactly once), so we may assume that the band intersects every essential sphere in the exterior of $L_\alpha = K_1 \cup K_2$ more than once. Let $W = \eta(K_1 \cup K_2 \cup b)$ where b is the band. (Note the ambiguity associated with the letter 'b' in this context.) Let α be a disc in $\eta(b)$ intersected once transversally by the core of b. Let β be a disc intersecting α once and which is "parallel" to the cocore of the band so that $L_\beta = K_1 \#_b K_2$. Since the band sum is not a connected sum, $\partial W - \partial\alpha$ is incompressible in $S^3 - \mathring{W}$ (Lemma 8.2). Applying Corollary 8.4, we produce a minimal genus Seifert surface $\overline{Q}$ for L_β which is disjoint from $\overline{\beta}$, the cocore of the band. The proof now proceeds as in [**G2**] and [**S3**]. $\square$

Superadditivity of genus under band sum provides a more interesting estimate of the genus of a knot L_β obtained by a rational tangle replacement on a split link than does Corollary 8.4. To see this, notice that the rational tangle replacement on a split link can be seen as a band sum of knots K_1 and K_2 with a 2–bridge knot K_3 inserted in the middle of the band. By moving the 2–bridge knot along the band so that it is close to K_2, we see that

$L_\beta = K_1 \#_b (K_3 \# K_2)$. Thus, by supperadditivity of genus under band sum,

$$\text{genus}(L_\beta) \geq \text{genus}(K_1) + \text{genus}(K_3) + \text{genus}(K_2).$$

I believe that the result of Corollary 8.4 for L_β an unknot is genuinely new. Similar to the previous case, this result can be interpreted as a result about attaching a band to a 2–bridge knot or link. However, not every such band attachment can be described as a rational tangle replacement on the unknot. The application of the Band Sum Genus theorem to rational tangle replacement on a split link is used in the next example to show that the possibility that $\overline{\beta}$ is isotopic into $\overline{Q}$ cannot be removed from Corollary 8.4.

EXAMPLE. Figure 8.2 depicts the diagram of a 9_{37} knot[1] L_β. The indicated rational tangle replacement converts L_β into a split link L_α. The rational tangle replacement has distance $d = 5$. In the diagram, it is not difficult to find a Seifert surface S for L_β consisting of an annulus and three twisted bands. Two of the bands have one half twist each and the third has three half twists. Thus, $-\chi(S) = 3$ and genus$(S) = 2$. L_β is the band sum of the unknot with a figure eight knot. The band is not disjoint from Seifert surfaces for the unknot and the figure eight knot. Hence, by Corollary 8.5, $\overline{Q}$ is a minimal genus Seifert surface for L_β. It is easy to see that $\overline{\beta}$ is isotopic into $\overline{Q}$.

REMARK. Scharlemann and Thompson [**ST2**] have shown that, in many cases, a tunnel for a tunnel number 1 knot can be isotoped and slid to lie in a minimal genus Seifert surface for the knot. Since tunnel number 1 knots

[1]Thanks to Jiho Kim's KnotSketcher and Charles Livingston's KnotFinder for help with this calculation.

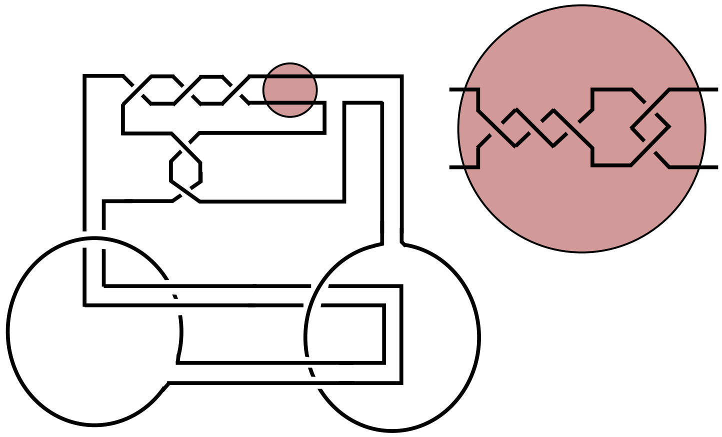

FIGURE 8.2. The knot L_β and a rational tangle replacement.

are those knots which are obtained by boring the unknot or unlink using an unknotted handlebody, perhaps the first possible conclusion of Corollary 8.4 points to a more general phenomenon.

8.3. Planar Surfaces, Punctured Tori, and Rational Tangle Replacement

We now use sutured manifold techniques to study planar surfaces and punctured tori in the exterior of a knot or link L_β obtained by rational tangle replacement on L_α.

THEOREM 8.6. *Suppose that L_β is a knot or link obtained by a rational tangle replacement of distance $d \geq 1$ on the knot or link L_α. Suppose that either L_α is a knot or that $\partial W - \partial \alpha$ does not compress in N. Suppose also that L_α is a split link, or does not contain a minimal genus Seifert surface disjoint from $\overline{\alpha}$. Then, if L_β has an essential properly embedded meridional planar surface with m boundary components, it contains such a surface $\overline{Q}$*

with $|\partial\overline{Q}| \leq m$ *such that either* $\overline{Q}$ *is disjoint from* $\overline{\beta}$ *or*

$$|\overline{Q} \cap \overline{\beta}|(d-1) \leq |\partial\overline{Q}| - 2$$

PROOF. Since either $\partial W - \partial\alpha$ is incompressible in N or α is non-separating, by Lemma 8.2, $(N, \gamma \cup a)$ is a taut sutured manifold. If L_β were a split link or unknot, by the first sutured manifold theorem, L_α would not be a split link and would have a minimal genus Seifert surface disjoint from $\overline{\alpha}$, a contradiction. Hence L_β is not a split link or unknot.

Use Corollary 5.2 to obtain the connected planar surface $\overline{Q} \subset N[b]$ and assume that $\overline{Q}$ is not disjoint from $\overline{\beta}$. Since $\overline{Q}$ is connected and has euler characteristic not lower than our original planar surface, $|\partial\overline{Q}| \leq m$. The boundary of $\overline{Q}$ is meridional, by construction, since each arc of $a - b$ is meridional. Since $\overline{Q}$ is, by assumption, not disjoint from $\overline{\beta}$, $\widetilde{q} > 0$ and there is no a–boundary compressing disc for Q.

By the first sutured manifold theorem and Lemma 7.1, we conclude that $K(\overline{Q}) \leq -2\chi(\overline{Q})$. Since $\partial\overline{Q}$ is disjoint from $a \cup \gamma$, if L_α is a link we obtain:

$$2q(d-1) + 2q^*(2d-1) \leq -2\chi(\overline{Q}).$$

If L_α is a knot, then

$$2(q + 2q^*)(d-1) \leq -2\chi(\overline{Q}).$$

Since $4q^*(d-1) \leq 2q^*(2d-1)$, we may conclude (whether or not α is separating) that $2(q+2q^*)(d-1) \leq -2\chi(\overline{Q})$. $\overline{Q}$ is a planar surface with $|\partial\overline{Q}|$ boundary components, implying that $-2\chi(\overline{Q}) = 2|\partial\overline{Q}| - 4$. Plugging into our inequality and dividing by two, we obtain

$$(q+2q^*)(d-1) \leq |\partial \overline{Q}| - 2.$$

A slight isotopy pushing the discs in $\overline{Q}$ with boundary parallel to b^* converts each such disc to two discs each with boundary parallel to b. Hence, after the isotopy $|\overline{Q} \cap \overline{\beta}| = q + 2q^*$. Consequently,

$$|\overline{Q} \cap \overline{\beta}|(d-1) \leq |\partial \overline{Q}| - 2$$

as desired. □

A crossing change or generalized crossing change of a knot K is achieved by choosing a disc $D \subset S^3$ which is pierced twice by K with opposite sign and by performing a $\pm 1/n$ Dehn-surgery on ∂D with $n \in \mathbb{N}$. If $n = 1$, the new knot is obtained by changing the crossing of K. It is easy to see that a generalized crossing change can be achieved by rational tangle replacement of distance $d = 2n$.

COROLLARY 8.7 (Scharlemann [**S1**], Scharlemann and Thompson [**ST1**]). *No generalized crossing change on a composite knot will produce the unknot.*

PROOF. Suppose that $L_\beta = K_1 \# K_2$ is an unknotting number one knot with K_1 and K_2 non-trivial knots. Let D be a crossing disc for L_β such that $\pm 1/n$ surgery on ∂D converts L_β to the unknot L_α. Let $W = \eta(L_\beta \cup D)$ and notice that L_α can be obtained from L_β by a rational tangle replacement of distance $d = 2n$. Notice that α is non-separating.

Apply Theorem 8.6 beginning with an essential meridional annulus in $S^3[\beta]$. The surface $\overline{Q}$ is then either an essential annulus or an essential disc. Since L_β is not the unknot, $\overline{Q}$ is an annulus. If it were disjoint from $\overline{\beta}$, the crossing change would be a crossing change on either K_1 or K_2 and so would not convert L_β into the unknot. The inequality $|\overline{Q} \cap \overline{\beta}|(d-1) \leq |\partial\overline{Q}| - 2$ becomes

$$0 \leq |\overline{Q} \cap \overline{\beta}|(d-1) \leq 0,$$

implying that $\overline{Q}$ is disjoint from $\eta(b)$ after all. This contradiction shows that L_β cannot be a composite unknotting number one knot. $\square$

In fact, in the spirit of Theorem 7.8, the results of this paper can be used to prove a (weak) version of Scharlemann and Thompson [**ST1**] about changing a crossing on a satellite knot. (See Section 8.4). As in Scharlemann and Thompson's work, this can be used to give another proof that unknotting number one knots are prime. The previous corollary, however, is an easier proof of that fact.

If a non-trivial surgery on a hyperbolic knot or link $L_\beta \subset S^3$ produces a manifold containing an essential sphere or torus, it is easy to show that the exterior of L_β contains an essential planar surface or punctured torus. The remaining theorem examines the possibilities for such surfaces in the exterior of a knot L_β obtained by rational tangle replacement on a split link or knot without a minimal genus Seifert surface disjoint from the boring arc.

THEOREM 8.8. *Suppose that L_β is a knot or link obtained by rational tangle replacement of distance $d \geq 1$ on a knot or link L_α using handlebody W.*

Suppose either that α is non-separating or that $\partial W - \partial\alpha$ is incompressible in N. Suppose also that L_α is a split link or does not have a minimal genus Seifert surface disjoint from $\overline{\alpha}$. Then, if L_β contains an essential planar surface or punctured torus in its exterior, there is such a surface $\overline{Q}$ satisfying one of the following:

(1) L_β is a link and $\partial\overline{Q}$ is disjoint from some component of L_β.
(2) $\overline{Q}$ is disjoint from $\overline{\beta}$ and $\overline{\beta}$ is isotopic into $\overline{Q}$.
(3) $\overline{Q}$ has meridional boundary on some component of L_β
(4) L_β and L_α are both links, $d = 2$, and $\overline{Q}$ is a punctured torus disjoint from $\overline{\beta}$ with integer slope on both components of $\partial S^3[\beta]$.
(5) L_β is a link, L_α is a knot, $d \leq 2$, and $\overline{Q}$ is a planar surface.
(6) L_β is a link, L_α is a knot, $d \leq 3$, and $\overline{Q}$ is a punctured torus.
(7) L_β is a knot, L_α is a link, $d = 1$, $\overline{Q}$ is a punctured torus with $\partial\overline{Q}$ having integer slope.
(8) L_β and L_α are both knots, $d = 1$ and $\overline{Q}$ is a planar surface.
(9) L_β and L_α are both knots, $d \leq 2$ and $\overline{Q}$ is a punctured torus.

PROOF. Since α is non-separating or $\partial W - \partial\alpha$ is incompressible in N, Lemma 8.2 implies that $(N, \gamma \cup a)$ is taut. By hypothesis, there is an essential planar surface or punctured torus in $S^3[\beta]$. Apply Corollary 5.2 to obtain a connected surface $\overline{Q}$. $\overline{Q}$ is a planar surface or a punctured torus. Assume that none of options (1), (2), or (3) occur. By Lemma 8.3, there is no a–boundary compressing disc for $\overline{Q}$. If $\overline{Q}$ is disjoint from a then it is disjoint from all meridional arcs of $a - b$ and so must have meridional boundary or must be disjoint from some component of $\partial S^3[\beta]$, contradicting our denial of (1) and (3). Hence, we may apply the first sutured manifold theorem to conclude that $-2\chi(\overline{Q}) \geq K(\overline{Q})$. Let $s = 2$ if $\overline{Q}$ is a planar surface and let $s = 0$ if $\overline{Q}$ is a punctured torus. We now consider the possibilites for α and β. We use the notation and results of the Tangle Calculation Lemmas.

Case 1: β and α are both separating. In this case, notice that $d \geq 2$. Since $-2\chi(\overline{Q}) = -2s + 2(|\partial_1\overline{Q}| + |\partial_2\overline{Q}|)$ we have

$$-2s + 2(|\partial_1\overline{Q}| + |\partial_2\overline{Q}|) \geq 2q(d-1) + d(|\partial_1\overline{Q}|n_1 + |\partial_2\overline{Q}|n_2).$$

Rearrange this to obtain

$$-2s \geq 2q(d-1) + |\partial_1\overline{Q}|(dn_1 - 2) + |\partial_2\overline{Q}|(dn_2 - 2).$$

If $\overline{Q}$ is a planar surface, then we must have either $dn_1 < 2$ or $dn_2 < 2$. Since d, n_1, and n_2 are all non-zero by hypothesis, we contradict the observation that $d \geq 2$. Hence $\overline{Q}$ is not a planar surface.

If $\overline{Q}$ is a punctured torus, then we must have $dn_1 \leq 2$ and $dn_2 \leq 2$. Since $d \geq 2$, we must have $d = 2$ and $n_1 = n_2 = 1$. This is conclusion (4).

Case 2: β is separating and α is non-separating. We have

$$-2s + 2(|\partial_1\overline{Q}| + |\partial_2\overline{Q}|) \geq 2q(d-1) + (d-1)(|\partial_1\overline{Q}|n_1 + |\partial_2\overline{Q}|n_2).$$

Rearranging, we obtain

$$-2s \geq 2q(d-1) + |\partial_1\overline{Q}|((d-1)n_1 - 2) + |\partial_2\overline{Q}|((d-1)n_2 - 2).$$

Therefore, we have $(d-1)n_1 \leq 2$ or $(d-1)n_2 \leq 2$. If $\overline{Q}$ is a planar surface the inequalities are strict. This produces conclusion (5). Otherwise, we obtain conclusion (6).

Case 3: β is non-separating and α is separating. Now we have,

$$-2s + 2|\partial\overline{Q}| \geq 2q(d-1) + 2q^*(2d-1) + 2d|\partial\overline{Q}|n.$$

Rearranging we find

$$-2s \geq 2q(d-1) + 2q^*(2d-1) + |\partial\overline{Q}|(2dn-2).$$

Since d, n, and $|\partial\overline{Q}|$ are all positive, $s=0$ and $d=n=1$. This is conclusion (7).

Case 4: β and α are both non-separating. Finally, we have

$$-2s \geq 2(q+2q^*)(d-1) + 2|\partial\overline{Q}|((d-1)n-1)$$

If $s=2$, then $(d-1)n<1$ implying $d=1$. If $s=0$, then $(d-1)n \leq 1$. This implies $d \leq 2$. These are conclusions (8) and (9). $\square$

8.4. More Classical Results

Our final look at rational tangle replacement will be to provide new proofs of several results of Eudave-Muñoz and others. The introduction listed six theorems, of which we can reprove five. The original proofs of all six theorems relied heavily on very complicated combinatorial arguments. In some sense, the arguments given here are more complicated in that they rely on sutured manifold theory and the additional work of this thesis. The present arguments, however, have the advantage of unifying most of the previous results. There is the additional hope that new proofs of the classical results can pave the way for proofs of related unsolved problems in knot theory. We begin by proving the five theorems just mentioned; they will be repeated here for the convenience of the reader. Weakened forms of two more theorems of Eudave-Muñoz will be given new proofs subsequently. All of the proofs in this section are very similar to prior proofs. We give them for completeness and to demonstrate the relative ease (given the machinery) with which they can be proven.

THEOREM (EM 2). *If (B,τ) is prime, if L_α is a split link, and if L_β is composite then $d(\alpha,\beta) \leq 1$.*

PROOF. Since (B,τ) is prime, by Lemma 8.2, $(S^3[\alpha],\gamma)$ is $\overline{\alpha}$–taut and $(N,\gamma\cup a)$ is $\varnothing$–taut. Let $\overline{R}$ be an essential meridional annulus in $S^3[\beta]$. Apply Corollary 5.2 to obtain an essential annulus or disc $\overline{Q}$. If $\overline{Q}$ is a disc, it must have meridional boundary since it was obtained by an a–boundary compression of a meridional annulus and all arcs of $a-b$ are meridional. This cannot occur since S^3 has no non-separating 2–spheres. Hence, $\overline{Q}$ is an essential meridional annulus. If $\overline{Q}$ were disjoint from $\overline{\beta}$ it would be contained in (B,τ). The boundary of $\overline{Q}$, in that case, must be on a single string of τ and so τ would contain a local knot, contradicting the assumption that (B,τ) is prime. Thus $\widetilde{q}(\overline{Q}) > 0$. We may now apply either of the first or second sutured manifold theorems to conclude that $-2\chi(\overline{Q}) \geq K(\overline{Q}) \geq 0$.

Since $\overline{Q}$ is an annulus, we have $K(\overline{Q}) = 0$ which by the Tangle Calculations implies that $d \leq 1$. $\square$

THEOREM (EM 3). *If (B, τ) is any tangle and if L_α and L_β are split links, then $r_\alpha = r_\beta$.*

PROOF. It suffices to show that α and β are disjoint. Suppose first that both $\partial N - \partial\alpha$ and $\partial N - \partial\beta$ are compressible in $N = B - \mathring{\eta}(\tau)$. By Lemma 8.1, since α is separating, $a = \partial\alpha$ compresses in N. That is, there is a disc D_a in B with boundary a separating the strings of τ. Similarly, there is a disc D_b in B with boundary $b = \partial\beta$ separating the strings of τ. An easy innermost disc/outermost arc argument shows that D_a and D_b are isotopic. In particular, a and b are isotopic in $\partial B - \tau$ which implies that $r_\alpha = r_\beta$.

Thus we may assume, without loss of generality, that $\partial W - \partial\alpha$ is not compressible in N. Let $\overline{R}$ be an essential sphere in $S^3[\beta]$ and apply Corollary 5.2 to obtain an essential sphere or disc $\overline{Q}$. Since $a - b$ consists of meridional arcs, $\overline{Q}$ is not disjoint from $\eta(a)$. If $\overline{Q}$ is a disc disjoint from $\overline{\beta}$, there is no a–boundary compressing disc for $\overline{Q}$. If $\overline{Q}$ is a sphere, $\widetilde{q} > 0$. Thus, we may apply either the first or second sutured manifold theorem to conclude that $S^3[\alpha]$ is irreducible or that α and β are disjoint. If the latter is true, $r_\alpha = r_\beta$. $\square$

THEOREM (BS 4). *If (B, τ) is a prime tangle and if L_α and L_β are both unknots, then $r_\alpha = r_\beta$.*

PROOF. Suppose that $r_\alpha \neq r_\beta$ so that $d \geq 1$. As in the proof of Theorem (EM 2), $(N, \gamma \cup a)$ is $\varnothing$–taut. Let $\overline{R}$ be an essential disc in $S^3[\beta]$. Let $\overline{Q}$

be a disc obtained by an application of Corollary 5.2. As in the proof of Theorem (EM 3), $\overline{Q}$ is not disjoint from a and if it is disjoint from $\overline{\beta}$ there is no a–boundary compressing disc. The first sutured manifold theorem then guarantees that there is a minimal genus Seifert surface for L_α which is disjoint from $\overline{\alpha}$. If L_α is an unknot, this surface S is a disc. Consider $S \cap \partial B$. Since $S \cap \overline{\alpha} = \varnothing$, the intersection consists of two arcs. If the outermost discs of $S - \partial B$ were located in B, at least one of the strands of τ is isotopic into ∂B in the complement of the other strand. Thus, there would be a disc separating the strands of τ, contrary to the hypothesis that (B, τ) is prime. Thus, the two strands of τ are parallel in B by the middle rectangle of $S - \partial B$. Reversing the roles of α and β in the preceding argument, we see that the two strands of τ are isotopic by a disc disjoint from $\partial \beta$. Consequently b and a are parallel in $\partial B - \tau$. Hence, α and β are parallel, contrary to our initial assumption that $d \geq 1$. $\square$

THEOREM (S 5). *If (B, τ) is any tangle and L_β is a trivial knot and L_α a split link then (B, τ) is a rational tangle and $d \leq 1$.*

PROOF. Notice first that (B, τ) can have no local knots since L_β is the unknot. Thus, if $\partial W - \partial \alpha$ is compressible in N, (B, τ) is a rational tangle with trivializing disc having boundary $\partial \alpha$ (Lemma 8.1).

Suppose that $\partial W - \partial \alpha$ is incompressible in N. Use Corollary 5.2 to choose a disc $\overline{Q}$ in $S^3[\beta]$. By the first or second sutured manifold theorems, $S^3[\alpha]$ is irreducible, a contradiction.

Thus, (B, τ) is a rational tangle. It remains to prove that $d \leq 1$. Since L_β is the unknot, a double-branched cover of S^3 with branch set L_β is S^3. The preimage $\widetilde{B}$ of B is an unknotted solid torus. There is a correspondence between rational tangle replacement and Dehn-surgery in the double-branched cover. Replacing (B', r_β) with (B', r_α) converts the double-branched cover to a lens space, S^3 or $S^1 \times S^2$. In the double branched cover, the Dehn surgery is achieved by making a curve in $\partial\widetilde{B}$ which intersects a meridian of $\widetilde{B}$ d times bound a disc in the complementary solid torus. Since L_α is a split link, the double branched cover of S^3 over L_α is reducible. Thus, it must be $S^1 \times S^2$ and d must be one, as desired. $\square$

REMARK. In the proof of (S 5), note that even without proving $d \leq 1$, we have provided a new proof of Scharlemann's band sum theorem [**S1**]: If $K = K_1 \#_b K_2$ is the unknot then the band sum is the connected sum of unknots. To see this note that W is $\eta(K_1 \cup K_2 \cup b)$ where b is the band. The tangle (B, τ) is $(S^3 - \mathring{\eta}(b), (K_1 \cup K_2) - \mathring{\eta}(b))$. Since $\partial\beta$ is a loop which encircles the band, $\partial\beta$ only bounds a disc in (B, τ) when the band sum is a connected sum and K_1 and K_2 are unknots.

THEOREM (EM 6). *If (B, τ) is prime and L_β is a composite knot or link and L_α is the unknot, then $d \leq 1$.*

PROOF. Suppose $d \geq 1$. First, suppose that there is no essential disc in $S^3[\alpha]$ which is disjoint from $\overline{\alpha}$. Let $\overline{Q}$ be the result of applying Corollary 5.2 to an essential meridional annulus in $S^3[\beta]$. Since L_β is not the unknot, $\overline{Q}$ is also a meridional annulus. It cannot be contained in B, since (B, τ) is prime. By Theorem 8.6,

$$|\overline{Q} \cap \overline{\beta}|(d-1) \leq 0.$$

Hence, $d = 1$, as desired.

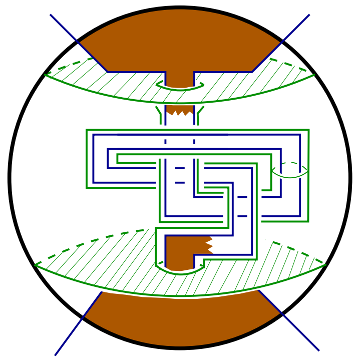

FIGURE 8.3. Enlarging a disc of parallelism to an annulus.

Suppose, therefore, that $S^3[\alpha]$ contains an essential disc which is disjoint from $\overline{\alpha}$. As in (BS 4), the two strands of τ must be parallel. Let A be an annulus in $B - \tau$ made by doubling and slightly enlarging the disc of parallelism (see Figure 8.3). Since (B, τ) is prime, A is an essential annulus in $B - \tau$. Let $D_\pm$ be the two discs in ∂B with boundary ∂A and which contain $\partial \tau$. Create a torus $T = A \cup (\partial B - (D_+ \cup D_-))$. By an isotopy, we may assume that T and $\overline{Q}$ are disjoint or intersect in circles which are essential in both. If the former, then $\overline{Q} \subset B$ and we contradict the assumption that (B, τ) has no local knots. Since every $S^2 \subset S^3$ separates, there are two circles of $\overline{Q} \cap T$ which are outermost on $\overline{Q}$. (That is, the circles adjacent to $\partial \overline{Q}$ are distinct.) By cutting and pasting T and $\overline{Q}$ along those circles we can turn T into two meridional annuli. Continuing in this way to eliminate in pairs circles of $\overline{Q} \cap T$ we turn T into a collection of meridional annuli disjoint from the original torus T. Equivalently, we see that the torus T was formed by tubing together meridional annuli. Hence, there is an essential meridional annulus for L_β contained in B (since (B', r_β) is a rational tangle). This, however, contradicts the assumption that (B,τ) was prime. $\square$

We now prove weak versions of two more theorems of Eudave-Muñoz [**EM4**]. There are several reasons why these versions are weaker.

(1) We are not allowing τ to contain circles in addition to the two arcs. Our methods could easily be extended to take care of this situation, since the main theorems allow ∂N to have additional torus components.
(2) It is not *a priori* clear that the conversion of a surface $\overline{R}$ to a surface $\overline{Q}$ of the same topological type using Corollary 5.2 can always be accomplished by an isotopy. A closer analysis of the methods of that theorem might show that in all situations of interest (e.g. if $S^3[\beta]$ does not contain an essential disc) it can be.
(3) In several of the possible conclusions of the first theorem we leave out certain very strong statements. More will be said about this after the proof.

Finally, a knot or link is **doubly composite** if there is an essential Conway sphere dividing the knot or link into two prime tangles.

THEOREM 8.9 (Eudave-Muñoz). *Suppose that (B,τ) is prime and that L_β is doubly composite. Let L_α be obtained by a rational tangle replacement of distance d from L_β. Then one of the following holds*

(1) There is an essential Conway sphere for L_β contained in B.
(2) $d \leq 1$

(3) *$S^3 - L_\alpha$ is irreducible and there is a minimal genus Seifert surface for L_α which is disjoint from $\overline{\alpha}$ (i.e. intersects ∂B in two arcs only).*

(4) *$d = 3$ and there is an essential Conway sphere for L_β intersecting ∂B in exactly one circle.*

(5) *$d = 2$ and there is an essential Conway sphere for L_β intersecting ∂B in two circles.*

(6) *$d = 2$ and there is an essential Conway sphere for L_β intersecting ∂B in one circle.*

PROOF. Since (B, τ) is prime, by Lemma 8.2, $(N, \gamma \cup a)$ is taut. Let $\overline{R}$ be an essential Conway sphere for L_β dividing L_β into two prime tangles. Apply Corollary 5.2 to obtain an essential connected meridional planar surface $\overline{Q}$ with no more boundary components than $\overline{R}$. (The surface is guaranteed to be meridional, since all arcs of $a - b$ are meridional.) Since every 2–sphere in S^3 separates, $\overline{Q}$ has either 2 or 4 boundary components. If $\overline{Q}$ were an annulus it would be disjoint from the Conway sphere and would contradict the assumption that the Conway sphere divided L_β into two prime tangles. Hence, $\overline{Q}$ is a Conway sphere. Assume that the first two possibilities do not occur, so that $\overline{Q} \cap \overline{\beta} \neq 0$ and $d \geq 2$. From the Tangle Equations, we see that $K(\overline{Q}) \geq 2(d-1)(q+2q^*)$. Also, $-2\chi(\overline{Q}) = 4$. If $-2\chi(\overline{q}) < K(\overline{Q})$ then by the first sutured manifold theorem and Lemma 7.1 there is a Seifert surface for L_α disjoint from $\overline{\alpha}$, and so (3) holds. Clearly this occurs if $d \geq 3$ since $q + 2q* \geq 1$. We may, therefore, assume that $-2\chi(\overline{Q}) \geq K(\overline{Q})$. Hence:

$$2 \geq (d-1)(q+2q^*).$$

Suppose that $d = 3$. Then $1 \geq q + 2q^* \geq 1$, implying conclusion (4). If $d = 2$, then $2 \geq q + 2q^* \geq 1$, implying that if $q^* > 0$ then $q = 0$ and $q^* = 1$. If $q^* = 1$ then a slight isotopy of $\overline{Q}$ moves the intersection curve of $\overline{Q}$ parallel to b^* to two intersection curves both parallel to b. Thus (5) or (6) holds. $\square$

REMARK. Now that the theorem has been proven it's worth remarking on the third much more significant reason why our version is weaker than Eudave-Muñoz's. Eudave-Muñoz is able to conclude that in cases (4) and (5), $S^3[\alpha]$ is irreducible and there is a minimal genus Seifert surface for L_α which is always intersected in the same direction by $\overline{\alpha}$. In these cases, since $-2\chi(\overline{Q}) \geq K(\overline{Q})$ the methods of this paper cannot give these conclusions. However, these conclusions are very similar to the sorts of results given by an application of the second sutured manifold theorem. This suggests that perhaps the inequality $-2\chi(\overline{Q}) \geq K(\overline{Q})$ in the second sutured manifold theorem could be made strict. Also, since these conclusions are different than what would be given by the first sutured manifold theorem, perhaps the inequality must always be strict in that theorem. That is, perhaps the second sutured manifold theorem can be strengthed in a way that the first one can't be. Eudave-Muñoz first proves a slightly stronger version of the theorem above (using different sutured manifold theorems) and then gives a separate lengthy combinatorial argument to obtain the much stronger version of conclusions (4) and (5).

We may also obtain a version of [**EM2**, Theorem 1.4]. This theorem generalizes a theorem of Scharlemann and Thompson [**ST1**]. It can be used to give another proof that unknotting number one knots are prime.

THEOREM 8.10 (Eudave-Muñoz). *Suppose that* (B, τ) *is prime and* L_β *is a satellite knot or link. Then one of the following holds:*

- *There is an essential torus in the exterior of* L_β *which is contained in* B.
- $d \leq 1$.
- L_α *has a minimal genus Seifert surface* S *for which* $S \cap \partial B$ *consists of two arcs (i.e.* S *is disjoint from* $\overline{\alpha}$*).*

PROOF. Since (B, τ) is prime, by Lemma 8.2, $(N, \gamma \cup a)$ is taut. Let $\overline{R}$ be an essential torus in $S^3[\beta]$ and apply Corollary 5.2 to obtain a surface $\overline{Q}$. Since L_α and L_β are related by a rational tangle replacement, $\overline{Q}$ is either an essential meridional annulus or an essential torus. $\overline{Q}$ cannot be an essential meridional annulus disjoint from $\overline{\beta}$ since (B, τ) is prime. If it is a torus disjoint from $\overline{\beta}$ then it is contained in B, the first conclusion. Thus, we may assume that $\widetilde{q} > 0$. If $0 = -2\chi(\overline{Q}) < K(\overline{Q})$, by the first sutured manifold theorem, the third possible conclusion holds. Suppose, therefore, that $0 \geq K(\overline{Q})$. Thus,

$$0 = K(\overline{Q}) \geq 2(d-1)(q + 2q^*) \geq 0$$

and $\widetilde{q} > 0$, we conclude that $d = 1$. This is the second possible conclusion.

□

CHAPTER 9

Intersections of $\varnothing$–taut Surfaces

The previous applications have shown that in many situations the first sutured manifold theorem is more useful than the second sutured manifold theorem. There are, however, two situations when the second is more useful. The first situation is when there is in $N[a]$ an essential separating sphere intersected twice by $\overline{\alpha}$. The second is when we wish to study a homology class in $H_2(N[a], \partial N[a])$ which is not represented by a surface disjoint from $\overline{\alpha}$. The propositions of this section consist of observations which can dramatically simplify the combinatorics of such a situation. Let N be a compact, orientable 3–manifold with $F \subset \partial M$ a genus 2 boundary component. Let $a, b \subset F$ be essential curves which cannot be isotoped to be disjoint and suppose that $(N[a], \gamma)$ is $\overline{\alpha}$–taut, as in Chapter 4.

9.1. Intersection Graphs

PROPOSITION 9.1. *Let $(N[a], \gamma)$ and b be as above and suppose that $z \in H_2(N[a], \partial N[a])$ is a non-trivial homology class. Suppose that $N[a]$ does not contain an essential disc disjoint from $\overline{\alpha}$. Then z is represented by an embedded conditioned $\overline{\alpha}$–taut surface $\overline{P}$. Furthermore, for any such $\overline{P}$, either $\overline{P}$ is disjoint from $\overline{\alpha}$ or $P = \overline{P} \cap N$ contains no b–boundary compressing discs or b–torsion $2g$–gons.*

PROOF. Let $\overline{P}$ be a conditioned $\overline{\alpha}$–taut surface. (Such a surface is guaranteed to exist by Theorem 2.1.) Suppose that $\overline{P}$ is not disjoint from $\overline{\alpha}$. Recall from the definition of "$\overline{\alpha}$–taut" that $\overline{\alpha}$ intersects $\overline{P}$ always with the same sign. Suppose that D is a b–torsion $2g$–gon for P. If $g = 1$, D is a b–boundary compressing disc for P. Let ε_i be the arcs $\partial D \cap F$. Let R be the rectangle containing the ε_i from the definition of b–torsion $2g$–gon. Suppose that the ends of R are on components of $\partial P - \partial\overline{P}$. The endpoints of the ε_i have signs arising from the intersection of ∂D with ∂P. Since $\overline{\alpha}$ always intersects $\overline{P}$ with the same sign an arc ε_i has the same sign of intersection at both its head and tail. Since the arcs are all parallel, all heads and tails of all the ε_i have the same sign of intersection. However, an arc of $\partial D \cap P$ must have opposite signs of intersection, arising as it does from the intersection of two surfaces. This implies that the head of some ε_i has a sign different from the tail of some ε_i, a contradiction. Hence, at least one end of R must lie on a component of $\partial\overline{P}$.

If one end of R is on $\partial P - \partial\overline{P}$ denote that component by a_1 and call the disc which it bounds in $\overline{P}$, α_1. If both ends of R are on $\partial\overline{P}$, let $\alpha_1 = \varnothing$. Attach R to $\overline{P} - \alpha_1$ creating a surface $\widetilde{P}$. The disc D is contained in N and, therefore, had interior disjoint from $\overline{\alpha}$. Compress $\widetilde{P}$ using D and continue to call the result $\widetilde{P}$.

An easy calculation shows that if $\alpha_1 \neq \varnothing$, then $\chi(\widetilde{P}) = \chi(\overline{P})$ but $|\overline{\alpha} \cap \widetilde{P}| = |\overline{\alpha} \cap \overline{P}| - 1$. Similarly, if $\alpha_1 = \varnothing$, then $-\chi(\widetilde{P}) = -\chi(\overline{P}) - 1$ and $|\overline{\alpha} \cap \widetilde{P}| = |\overline{\alpha} \cap \overline{P}|$. If $\chi_{\overline{\alpha}}(\overline{P}) \neq |\overline{\alpha} \cap \overline{P}| - \chi(\overline{P})$ then a component of $\overline{P}$ is a disc disjoint from $\overline{\alpha}$ or a sphere intersected by $\overline{\alpha}$ once. Either of these contradict our hypotheses on $N[a]$. Hence, $\chi_{\overline{\alpha}}(\overline{P}) = |\overline{\alpha} \cap \overline{P}| - \chi(\overline{P})$.

Similarly, $\chi_{\overline{\alpha}}(\widetilde{P}) = |\overline{\alpha} \cap \widetilde{P}| - \chi(\widetilde{P})$. Hence, $\chi_{\overline{\alpha}}(\widetilde{P}) = \chi_{\overline{\alpha}}(\overline{P}) - 1$. Since $\overline{\alpha}$ always intersects $\widetilde{P}$ with the same sign, $\overline{P}$ is not $\overline{\alpha}$–taut, a contradiction. Hence, there are no b–torsion $2g$–gons for P. □

REMARK. As Scharlemann notes in [**S5**], when a and b are non-separating it can be difficult to use combinatorial methods to analyze the intersection of surfaces in $N[a]$ and $N[b]$. The primary reason for this is the need to work with a^* and b^* boundary components on the surfaces. The previous proposition shows that when the surfaces in question are $\overline{\alpha}$–taut and $\overline{\beta}$–taut and not disjoint from $\overline{\alpha}$ and $\overline{\beta}$, respectively, there is no need to consider a^* and b^* curves.

The remainder of this section develops notation for studying the intersection graphs of such surfaces. Let $\overline{P} \subset N[a]$ be an $\overline{\alpha}$–taut surface and let $\overline{Q} \subset N[b]$ be a $\overline{\beta}$–taut surface. Suppose that $\overline{P}$ and $\overline{Q}$ are not disjoint from $\overline{\alpha}$ and $\overline{\beta}$ respectively. By Proposition 9.1 there is no b–torsion $2g$–gon for $P = \overline{P} \cap N$ and no a–torson $2g$–gon for $Q = \overline{Q} \cap N$.

In section 3.2, we defined intersection graphs between $\overline{Q}$ and a disc D. We now define, in a similar fashion, intersection graphs between $\overline{P}$ and $\overline{Q}$. Orient P (respectively, Q) so that all boundary components of $\partial P - \partial \overline{P}$ ($\partial Q - \partial \overline{Q}$, respectively) are parallel on $\eta(\overline{\alpha})$ ($\eta(\overline{\beta})$, respectively). The intersection of P and Q forms graphs Λ_α and Λ_β on $\overline{P}$ and $\overline{Q}$. A component of $\partial P - \partial \overline{P}$ or $\partial Q - \partial \overline{Q}$ is called an **interior boundary component**. The vertex of Λ_α or Λ_β to which it corresponds is called an **interior vertex**.

Label the components of $\partial Q \cap \eta(a)$ as $1 \dots \mu_Q$ and the components of $\partial P \cap \eta(b)$ as $1 \dots \mu_P$. The labels should be in order around $\eta(a)$ and $\eta(b)$.

An endpoint of an edge on an interior vertex of Λ_α corresponds to an arc of $\partial Q \cap \partial\eta(\alpha)$. Give the endpoint of the edge the label associated to that arc. Similarly, label all endpoints of edges on interior vertics of Λ_β. A **Scharlemann cycle** is a type of cycle which bounds a disc in $\overline{P}$ ($\overline{Q}$, respectively). The interior of the disc must be disjoint from Λ_α (Λ_β) and all of the vertices of the cycle must be interior vertices. Furthermore, the cycle can be oriented so that the tail end of each edge has the same label. This is the same notion of Scharlemann cycle as in Section 3.3.2, but adapted to the, possibly non-planar, surfaces $\overline{P}$ and $\overline{Q}$.

LEMMA 9.2. *There is no Scharlemann cycle in Λ_α or Λ_β.*

PROOF. Were there a trivial loop at an interior vertex or a Scharlemann cycle in Λ_α or Λ_β, the interior would be an a or b–torsion $2g$–gon, contradicting Proposition 9.1. □

The next lemma may be useful at some point in the future. It shows that if $\overline{P}$ is a disc, the presence of loops is strongly restricted:

LEMMA 9.3. *If $\overline{P}$ is a disc, then every loop in Λ_α is based at $\partial\overline{P}$.*

PROOF. Suppose that $\overline{P}$ is a disc and that there is a loop based at an interior vertex of Λ_α. A component X of the complement of the loop in $\overline{P}$ does not contain $\partial\overline{P}$. The loop is an x–cycle and Lemma 3.10 then guarantees the existence of a Scharlemann cycle in X, contrary to Lemma 9.2. □

9.2. When the exterior of W is anannular.

We conclude this section with an application to refilling meridians of a genus 2 handlebody whose exterior is irreducible, boundary-irreducible, and anannular. It is based on the ideas in [**SW**]. Suppose that M is the exterior of a link in S^3. Suppose that $W \subset M$ is a genus 2 handlebody embedded in M. Let $N = M - \mathring{W}$.

THEOREM 9.4. *Suppose that N is irreducible, boundary-irreducible and anannular. Suppose that α and β are non-separating meridians of W such that $\Delta > 0$. Suppose that neither $M[\alpha]$ nor $M[\beta]$ contain an essential disc or sphere. Suppose also that in $H_2(M[\alpha], \partial M)$ there is a homology class z_a which cannot be represented by a surface disjoint from $\overline{\alpha}$ and that in $H_2(M[\beta], \partial M)$ there is a homology class z_b which cannot be represented by a surface disjoint from $\overline{\beta}$. Then there is a $\varnothing$–taut surface $\overline{P} \subset M[\alpha]$ representing z_a intersecting $\overline{\alpha}$ p times and an $\varnothing$–taut surface $\overline{Q} \subset M[\beta]$ representing z_b intersecting $\overline{\beta}$ q times such that one of the following occurs:*

(1) $-2\chi(\overline{P}) \geq p(\mathscr{M}_b(a) - 2)$
(2) $-2\chi(\overline{Q}) \geq q(\mathscr{M}_a(b) - 2)$
(3) *All of the following occur:*
   - $\overline{Q}$ *is* $\overline{\beta}$*–taut*
   - $\overline{P}$ *is* $\overline{\alpha}$*–taut.*
   - $pq\Delta \leq 18(p - \chi(\overline{P}))(q - \chi(\overline{Q}))$
   - $\Delta < \frac{9}{2}\mathscr{M}_a(b)\mathscr{M}_b(a)$

PROOF. Notice that the right hand side of the inequalities in (1) and (2) are $K(\overline{P})$ and $K(\overline{Q})$ respectively. Choose a taut representative in $M[\beta]$ for z_b and apply Theorem 5.1, obtaining $\overline{Q}$. Since negative euler characteristic is not increased and $M[\beta]$ does not contain an essential disc or sphere, $\overline{Q}$ is also taut. If (1) holds, we are done, so assume that $-2\chi(\overline{Q}) < K(\overline{Q})$. Recall that $\overline{Q}$ is not disjoint from $\overline{\alpha}$. Apply the second sutured manifold theorem to obtain a surface $\overline{P} \subset M[\alpha]$ representing z_a. (The surface $\overline{P}$ is the surface S in the statement of that theorem.) $\overline{P}$ is both $\overline{\alpha}$–taut and $\varnothing$–taut. If (2) holds, we are done, so assume $-2\chi(\overline{P}) < K(\overline{P})$. Applying the second sutured manifold theorem again, with α and β reversed, we find a $\overline{\beta}$–taut and $\varnothing$–taut surface in $M[\beta]$ representing z_b. We may call this surface $\overline{Q}$, forgetting the previous one. Consider the the graphs formed by the intersection of P and Q; let Λ_α be the graph on $\overline{P}$ and Λ_β the graph on $\overline{Q}$. Lemma 9.2 assures us that there is no trivial loop based at an interior vertex of either graph.

LEMMA 9.5.

$$pq\Delta \leq 18(p - \chi(\overline{P}))(q - \chi(\overline{Q}))$$

PROOF OF LEMMA 9.5. By [**SW**, Lemma 2.1], if two edges of $P \cap Q$ are parallel in both Λ_α and Λ_β, there is an essential annulus in N, contrary to our assumption that N is anannular. The proof proceeds as in [**SW**].

Each interior boundary component of P intersects ∂Q, $q\Delta$ times. Thus $|\partial Q \cap \partial P| \geq pq\Delta$. Therefore, Λ_α and Λ_β each have at least $pq\Delta/2$ edges.

Claim: Λ_α has at least $\frac{pq\Delta}{6(p-\chi(\overline{P}))}$ mutually parallel edges.

This claim is similar to work in [**GLi**]. Let Λ' be the graph obtained by combining each set of parallel edges of Λ_α into a single edge. Since Λ' has no loops at interior vertices and no parallel edges, by applying the formula for the euler characteristic of a closed surface we obtain:

$$\begin{aligned}
\chi(\overline{P}) + |\partial\overline{P}| &= V - E + F \\
&\leq p + |\partial\overline{P}| - E + (2/3)E \\
&= p + |\partial\overline{P}| - (1/3)E
\end{aligned}$$

V, E, and F represent the number of vertices, edges, and faces of Λ'. Thus, $E \leq 3(p - \chi(\overline{P}))$. Let n be the largest number of mutually parallel edges in Λ_α. Then, since there are at least $pq\Delta/2$ edges in Λ_α, we have

$$pq\Delta/(2n) \leq E \leq 3(p - \chi(\overline{P})).$$

The claim follows.

A similar argument shows that if a graph in $\overline{Q}$ has more than $3(q - \chi(\overline{Q}))$ edges than two of them are parallel. Hence, since there are no mutually parallel edges in Λ_α and Λ_β we must have:

$$\frac{pq\Delta}{6(p - \chi(\overline{P}))} \leq 3(q - \chi(\overline{Q}))$$

whence the lemma and the first inequality of Conclusion 3 follow. □

We now proceed with the proof of the theorem. Since we are assuming that neither (1) nor (2) hold, we have

$$-\chi(\overline{P}) < K(\overline{P})/2 = p(\mathscr{M}_b(a) - 2)/2$$

$$-\chi(\overline{Q}) < K(\overline{Q})/2 = q(\mathscr{M}_a(b) - 2)/2$$

Plugging into the inequality from the lemma, we obtain

$$pq\Delta < 18pq\Big(1+\frac{\mathscr{M}_b(a)-2}{2}\Big)\Big(1+\frac{\mathscr{M}_a(b)-2}{2}\Big).$$

Since neither p nor q is zero, we divide and simplify to obtain:

$$\Delta < 9\mathscr{M}_b(a)\mathscr{M}_a(b)/2.$$

$\square$

REMARK. The point of the previous theorem is that, under the specified conditions, either we obtain a bound on the euler characteristic of surfaces representing the homology classes z_a or z_b or we obtain a restriction on the number of non-meridional arcs of $a-b$ and $b-a$. For example, suppose that discs α and β are chosen so that z_a is represented by a once-punctured torus, and so that $\mathscr{M}_b(a)=\mathscr{M}_a(b)=6$. Then $-2\chi(\overline{P})=2<4p=K(\overline{P})$. Then if z_b is also represented by a once punctured torus, we have $\Delta<162$. Since Δ is even, this implies $\Delta\leq 160$.

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