Math 253, Spring 2001, Final Exam, Solutions

1. F Consider the matrix $A=\begin{bmatrix}0&1\\0&0\end{bmatrix}$, with alg.mult.(0)=2 and geom.mult.(0)=1.

   T If $A\vec{v}=4\vec{v}$, then $A^4\vec{v}=4^4\vec{v}$.

   F Consider the zero matrix.

   F Consider the diagonal 10×10 matrix with diagonal entries 1,2,3,4,0,0,0,0,0,0.

   F Consider the matrix $A=\begin{bmatrix}0&1\\0&0\end{bmatrix}$

2. F $\det(-A)=-\det(A)$ for an $n\times n$ matrix A where n is odd.

   T Check the two properties of a linear transformation.

   T Since both A and A^{-1} are invertible, we have $\text{rank}(A)=\text{rank}\left(A^{-1}\right)=4$.

   F The zero matrix does not belong to V.

   T V is a line and can be described by two linear equations $ax+by+cz=0$ and $px+qy+rz=0$ (two intersecting planes). Thus $V=\ker\begin{bmatrix}a&b&c\\p&q&r\end{bmatrix}$.

3. If $A=\begin{bmatrix}-1&2\\4&1\end{bmatrix}$ and $S=\begin{bmatrix}1&1\\-1&2\end{bmatrix}$, then $B=S^{-1}AS=\begin{bmatrix}-3&0\\0&3\end{bmatrix}$.

4. $\text{rref}(A)=\begin{bmatrix}1&0&2&0&-1\\0&1&3&0&2\\0&0&0&1&3\\0&0&0&0&0\end{bmatrix}$ A. Basis of Image: $\begin{bmatrix}1\\0\\0\\1\end{bmatrix}, \begin{bmatrix}0\\1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\3\\0\end{bmatrix}$

   B. Basis of Kernel: $\begin{bmatrix}-2\\-3\\1\\0\\0\end{bmatrix}, \begin{bmatrix}1\\-2\\0\\-3\\1\end{bmatrix}$.

5. A. We want $f_A(\lambda)=\det\begin{bmatrix}\lambda-a & -1\\ -b & \lambda\end{bmatrix}=\lambda^2-a\lambda-b=(\lambda-2)(\lambda-3)=\lambda^2-5\lambda+6$

Thus $a=5,\quad b=-6$, and $A=\begin{bmatrix}5 & 1\\ -6 & 0\end{bmatrix}$.

B. Make sure that the discriminant of the characteristic polynomial, a^2+4b, is negative. For example, make $a=0$ and $b=-1$, so that $A=\begin{bmatrix}0 & 1\\ \ldots & \ldots\end{bmatrix}$ (a rotation)

6. A. Use a commutative diagram to find $A=\begin{bmatrix}1 & -2 & 2\\ 0 & 1 & -4\\ 0 & 0 & 1\end{bmatrix}$.

B. The only eigenvalue of A is 1 (with algebraic multiplicity 3), with associated eigenvectors $\begin{bmatrix}c\\0\\0\end{bmatrix}$, where c is a nonzero constant. . Converting these vectors into polynomials produces the eigenfunctions $f(x)=c$, where $c\neq 0$.

7. A is diagonalizable iff $\dim(E_1)=2$. Now $E_1=\ker\begin{bmatrix}0 & 1 & b\\ 0 & 2 & c\\ 0 & 0 & 0\end{bmatrix}=\ker\begin{bmatrix}0 & 1 & b\\ 0 & 0 & c-2b\\ 0 & 0 & 0\end{bmatrix}$.

This kernel is two-dimensional iff the second row is zero, that is, if $c=2b$.
Thus A is diagonalizable iff $c=2b$.

8. A. The eigenvalues for the given eigenvectors (in the given order) are $1, -1/2$, and $-1/2$. Now $\begin{bmatrix}6\\1\\2\end{bmatrix}=3\begin{bmatrix}1\\1\\1\end{bmatrix}+2\begin{bmatrix}1\\-1\\0\end{bmatrix}+1\begin{bmatrix}1\\0\\-1\end{bmatrix}$, so $\vec{x}(t)=3\begin{bmatrix}1\\1\\1\end{bmatrix}+2(-1/2)^t\begin{bmatrix}1\\-1\\0\end{bmatrix}+(-1/2)^t\begin{bmatrix}1\\0\\-1\end{bmatrix}$.

Thus $a(t)=3+3(-1/2)^t$, $b(t)=3-2(-1/2)^t$, $c(t)=3-(-1/2)^t$.

B. Since $n=365$ is odd, Benjamin will have the most money.

9. A. A straightforward computation shows that V consists of the matrices of the form

9. A. A straightforward computation shows that V consists of the matrices of the form $\begin{bmatrix} a & a \\ -d & d \end{bmatrix}$. Thus a basis of V is $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ -1 & 1 \end{bmatrix}$, and $\dim(V) = 2$.

B. The kernel of T is the space V discussed in part A, with $\dim(\ker T) = 2$. By the fundamental theorem, $\dim(\operatorname{im} T) = \dim\left(\mathbb{R}^{2\times 2}\right) - \dim(\ker T) = 4 - 2 = 2$.

10. A. $\ker(T) = \{f(x) : f'(x)\cdot x = 0\} = \{f(x) : f'(x) = 0\} = \{\text{constant functions}\}$. $\dim(\ker T) = 1$.

B. Any polynomial with a nonzero constant term, such as $f(x) = x^2 + 2$.

C. If $f(x) = a_0 + a_1x + a_2x^2 + ... + a_nx^n$, with $a_n \neq 0$, is an eigenfunction with eigenvalue λ, then $T(f(x)) = f'(x)\cdot x = a_1x + 2a_2x^2 + ... + na_nx^n = \lambda f(x)$ $= \lambda a_0 + \lambda a_1 x + \lambda a_2 x^2 + ... + \lambda a_n x^n$. Comparing coefficients, we find that $\lambda = n$ and $a_0 = a_1 = ... = a_{n-1} = 0$, so that $f(x) = a_nx^n$.

Thus the eigenvalues are all non-negative integers n, and the eigenfunctions associated with eigenvalue n are of the form $f(x) = cx^n$, where c is a nonzero constant.