A Solution to Problem 30

As indicated in *p-adic Numbers*, there is a solution in D. P. Parent's *Exercises in Number Theory.* Unfortunately, there are problems with Parent's solution. Here is my attempt to rewrite it.

Fact 0.0.1 *For each integer $M > 0$ there exists an n such that the partial sum*

$$2 + \frac{2^2}{2} + \frac{2^3}{3} + \frac{2^4}{4} + \cdots + \frac{2^n}{n}$$

is divisible by 2^M.

Problem 1 Can you give a direct proof of this fact?

First of all, a direct proof is *much* harder than the p-adic proof. Second, given the result it is fairly easy to estimate M; see problem 175 and its solution.

For an elementary proof, let $M > 0$ be given. We want to show that the numerator of

$$\sum_{n=1}^{N} \frac{2^n}{n}$$

is divisible by 2^M when N is large enough. We use congruence notation, so $a \equiv b \pmod{2^M}$ means that the difference $a - b$ is in $2^M\mathbb{Z}_2$.

Following Parent's idea, begin from the identity $(1-2)^{2^k} = 1$ for all $k \geq 1$. Expanding the binomial we get

$$\sum_{n=0}^{2^k} \binom{2^k}{n} (-2)^n = 1$$

and so, subtracting 1 from both sides,

$$\sum_{n=1}^{2^k} \binom{2^k}{n} (-1)^n 2^n = 0.$$

Dividing through by 2^k gives

$$\sum_{n=1}^{2^k} \binom{2^k}{n} (-1)^n \frac{2^n}{2^k} = 0.$$

Since binomial coefficients are integers, the terms with $n \geq M+k$ are certainly divisible by 2^M, so

$$\sum_{n=1}^{k+M-1} \binom{2^k}{n} (-1)^n \frac{2^n}{2^k} \equiv 0 \pmod{2^M}. \qquad (*)$$

The key observation now is that

$$\frac{1}{2^k}\binom{2^k}{n} = \frac{1}{2^k}\frac{2^k!}{n!(2^k-n)!} = \frac{(2^k-1)!}{n!(2^k-n)!}.$$

Now cancel $(2^k-n)!$ to get

$$\frac{1}{2^k}\binom{2^k}{n} = \frac{(2^k-1)(2^k-2)\cdots(2^k-n-1)}{n!}$$

If we expand the product in the denominator, most terms will be divisible by 2^k so

$$\frac{1}{2^k}\binom{2^k}{n} = \frac{2^k a(k,n)}{n!} + (-1)^{n-1}\frac{(n-1)!}{n!} = \frac{2^k}{n!}a(n,k) + (-1)^{n-1}\frac{1}{n},$$

where $a(n,k)$ is an integer.

Multiplying by $(-1)^n 2^n$ to get the general term in (*),

$$(-1)^n\frac{2^n}{2^k}\binom{2^k}{n} = (-1)^n\frac{2^{n+k}}{n!}a(n,k) - \frac{2^n}{n}.$$

This is the key trick, since it makes the term $2^n/n$ appear. Those, after all, are the ones we are trying to sum.

Plugging this new formula into (*) we get

$$\sum_{n=1}^{k+M-1}\left((-1)^n\frac{2^{n+k}}{n!}a(n,k) - \frac{2^n}{n}\right) \equiv 0 \pmod{2^M},$$

which, finally, gives us

$$\sum_{n=1}^{k+M-1}\frac{2^n}{n} \equiv \sum_{n=1}^{k+M-1}(-1)^n\frac{2^{n+k}}{n!}a(n,k) \pmod{2^M}.$$

Here M is given but $k \geq 1$ is arbitrary. Since $a(n,k) \in \mathbb{Z}$, it now suffices to choose k such that $2^{n+k}/n!$ is divisible by 2^M. Since $v_2(n!) < n$ (see Lemma 5.7.4, noting that here $p = 2$), we have $v_2(2^{n+k}/n!) > k$, so it suffices to choose $k \geq M$, i.e., to sum up to $k + M - 1 \geq M + M - 1 = 2M - 1$.

We conclude then, that if $N \geq 2M - 1$ then

$$\sum_{n=1}^{N}\frac{2^n}{n} \equiv 0 \pmod{2^M},$$

which proves the claim.