As promised, this is an itemized list of topics we've covered in Math 338 since the second midterm exam. As I stated in class, the final exam will be cumulative, with the coverage spread (approximately) evenly througout the entire course. Thus, when studying for your final, please concatenate this list of topics with the preview two to form a complete list of topics for the final exam. As before, please note that I consider the homework exercises and the everything I've covered in lecture to be the best source of practice (problems, proofs, etc). If you know how to approach each problem/exercise/proof, are able to work quickly and accurately, and understand the theory and methodology by which you have obtained a solution/proof, you should perform well on the exam.

Note: As promised, the lecture for the last day of class was recorded and can be found here: Lecture Recording, May 8th

Definitions:

The following list enumerates all the definitions you need to know (and by heart). In particular, you should make sure to know all quantifiers involved in the definitions and the order in which they appear. Also, for each definition, you should be able to come up with several examples satisfying the definition (and hopefully things that don't satisfy the definition).

1. You should know what it means for a function $f : X \to Y$ (for metric spaces (X, d_X) and (Y, d_Y)) to be Lipschitz and Hölder continuous of order $0 < \alpha \leq 1$.
2. For a function $f : [a, b] \to \mathbb{R}$, you should know what it means for f to be differentiable at a point $x \in [a, b]$.
3. You should know Definition 6.28 and why it makes sense to call it an average.
4. On a set X, you should know what it means for a sequence of complex-valued functions $\{f_n\}$ to converge pointwise to another function f on X. This is definition 7.1.
5. On a set X, you should know what it means for a sequence of complex-valued functions $\{f_n\}$ to converge uniformly to another function f on X. You should also know how this differs from pointwise convergence. This is definition 7.2.
6. On a set X, you should know what it means for a sequence of complex-valued functions to be "uniformly Cauchy" on X.
7. You should know Definition 7.5 and, in particular, what it means for a series of functions to converge pointwise and uniformly on a set X.
8. For an interval $I \subseteq \mathbb{R}$, you should know what it means for a function $f \in C^k(I; \mathbb{C})$ for $k = 0, 1, \dots$. In particular, you should know that $f \in C^0(I; \mathbb{C})$ means f is a continuous complex-valued function on I. $f \in C^1(I; \mathbb{R})$ means that f is a complex-valued function on I which is everywhere differentiable and its derivative, f', is a member of $C^0(I; \mathbb{R})$.
9. Given a set X, you should know that $B(X; \mathbb{C})$ is the set of bounded complex-valued function on X.
10. Given a metric space (X, d_X), $\mathscr{C}(X)$ is the set of bounded continuous complex-valued functions on X. We note, in the case that X is compact, $\mathscr{C}(X) = C^0(X; \mathbb{C})$.
11. You should know about the Weierstrass function (Example 7.5 in the course notes).
12. You should know that it means for a family of functions $\mathcal{A} \subseteq \mathscr{C}(X)$ to
    (a) be an algebra
    (b) separate points
    (c) vanish nowhere (or be nowhere vanishing)

(d) be self-adjoint

13. You should know what trigonometic polynomials are. As we discussed, these are polynomials of the form
$$\sum_{n=0}^{N} a_n \cos(nx) + b_n \sin(nx).$$
Thanks to Euler's formula, as we discussed, these can be equivalently written by
$$\sum_{n=-N}^{N} c_n e^{-inx}$$
which is much easier to work with. Could you verify that the collection of trigonometric polynomials is a self-ajoint algebra which separates points and vanishes nowhere?

14. You should know the what it means for a function $f : \mathbb{R} \to \mathbb{C}$ to be 2π periodic. You should also know (at least have a good understanding of the equivalence of 2π periodic functions and those on the unit circle S^1.

15. You should know how the spaces $R(\mathbb{T})$ and $C^k(\mathbb{T})$ are defined. I also mentioned $L^p(\mathbb{T})$ for $1 \leq p \leq \infty$. While you don't need to know how these spaces are defined, it is helpful to understand that these are all function spaces containing $R(\mathbb{T})$ that have certain norms/metrics (which are inequivalent!) defined on them. For $p = 2$, in particular, the norm comes from an inner product.

16. In line with the above, you should know the so-called L^2 inner product and norm are given respectively by
$$\langle f, g \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\overline{g(x)}\, dx$$
and
$$\|f\|_2 = \sqrt{\langle f, g \rangle} = \sqrt{\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|^2 dx}$$
for, at least, $f \in R(\mathbb{T})$.

17. You should know the *characters* $\{e_n\}$ defined by $e_n(x) = e^{inx}$ for $n \in \mathbb{Z}$ and $x \in \mathbb{R}$. You should know these form an orthonormal set relative to the L^2 inner product.

18. For $f \in R(\mathbb{T})$, the Fourier coefficients $\{\widehat{f}(n)\}$ are defined, for each $n \in \mathbb{Z}$, by
$$\widehat{f}(n) = \langle f, e_n \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx}\, dx.$$

19. For $f \in R(\mathbb{T})$ with Fourier coefficients $\{\widehat{f}(n)\}$, the Fourier series for f is the (formal) series
$$\sum_{n\in\mathbb{Z}} \widehat{f}(n) e^{inx}.$$
Of course, the convergence of this series is a big question we studied in the last week of class and you should be familiar with this. In particular, you should know that this question arose out of Fourier's solution (attempt) to the so-called Cauchy problem for the heat equation: Given initial "data" $u_0 : \mathbb{R} \to \mathbb{C}$ which is 2π-periodic, find a function $u(t, x)$ for which
$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2} & t > 0 \text{ and } x \in \mathbb{R} \\ u(0, x) = u_0(x) & x \in \mathbb{R} \end{cases}.$$

Fourier observed that, in the case that somehow one could express

$$u_0(x) = \sum_{n\in\mathbb{Z}} c_n e^{inx}$$

where the series converged in some reasonable way, then

$$u(t,x) = \sum_{n\in\mathbb{Z}} c_n e^{-n^2 t} e^{inx}$$

solves the above Cauchy problem for the heat equation. We also learned that, viewed through the lens of the L^2 inner product, the only reasonable choice of coefficients c_n above are the Fourier coefficients, i.e., $c_n = \widehat{u_0}(n)$. Hence, the problem is solved when, given a function u_0 whose Fourier series converges reasonably to it, the solution to the corresponding Cauchy problem for the heat equation is easily gotten by the above prescription.

Results (Theorems, propositions, lemmas, corollaries):

For the following results, unless otherwise mentioned, you should know the statement of the result precisely and have a really good idea of how they are proved – ideally, you should be able to follow/reproduce the proof.

1. You should know the fundamental theorem of calculus, Part I and Part II. These appear as Theorem 6.24 and Theorem 6.25 in the course notes. It should be mentioned that the proof of "Part II" relies on the mean value theorem, which we did not prove. I encourage you to read through the proof of the mean value Theorem(s) in Rudin's book, but you should know you will not be responsible for them.

2. You should know Theorem 7.3 in the course notes; this theorem shows that, in particular, any sequence of complex-valued functions that is uniformly Cauchy is uniformly convergent. What's important, in my eyes about this theorem is that it allows you to determine uniform convergence without knowing a limit function. This is particularly helpful in studying series of functions (See Corollary 7.6) wherein you almost never know what the sum of the series might be, but you can still easily decide convergence from the Cauchy condition.

3. You should know Corollary 7.4 and its proof.

4. You should know Corollary 7.6 and its proof.

5. You should know the Weierstrass M-test (Theorem 7.7) and its proof. You should also know how to apply the Weierstrass M test to determine convergence (we did much of this in the last few weeks of the semester).

6. You should know Theorem 7.8 and its proof (this is the $\epsilon/3$ proof!). You should also know that this theorem gives us one of our first interpretations of exchanging the order of limits.

7. You should know Theorem 7.9 and its proof (part of which was a problem on Midterm 2, as we discussed).

8. You should know Corollaries 7.10 and 7.11.

9. You should know the Weierstrass approximation theorem (Theorem 7.16) and the Stone Weierstrass theorem (Theorem 7.18). You do not need to know their proofs. Still, you should have a decent idea behind the proof of Theorem 7.16 (as integrating against certain polynomials). Note: You also gave a separate proof of this fact on the final homework set (Bernstein approximation).

10. You should know Theorem 8.26 and its proof.

11. One thing that comes out of the proof of 8.26 is Bessel's inequality (Theorem 8.27).

12. One immediate consequence of the divergence test and Bessel's inequality is the Riemann-Lebesgue Lemma, Corollary 8.28.

13. Theorem 8.21 (Parseval's Theorem) and its proof (which, notably makes use of the Stone-Weierstrass theorem).
14. You should know everything about the Example 8.2 with which obtained the value of the sum of the series
$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$
15. You should know Lemma 8.35 and its proof.
16. You should know Theorem 8.22 and its proof.
17. You should know Corollary 8.23 and its proof (note the content of this corollary is recaptured by Exercise 8.3, which you did on Homework 9).

Things you should know how to do:

1. For a sequence of of complex-valued functions $\{f_n\}$ on a set X (say $X = [a, b] \subseteq \mathbb{R}$), determine if the sequence converges pointwise or uniformly on X. You should also be able to do this for the corresponding series $\sum f_n$ provided the functions are reasonable enough.
2. Apply the Weierstrass M test to deduce uniform convergence of series.
3. For a uniformly convergent sequence or series, use results that allow you to exchange limits. For instance, if you know that a series $\sum_n f_n$ of complex-valued functions converges uniformly on $[0, 1]$, could you compute
$$\int_0^1 f(x)\,dx$$
where
$$f(x) = \sum_n f_n(x)?$$
4. Given a family of complex-valued functions $\mathcal{A} \subseteq \mathscr{C}(X)$, could you decide if $\mathcal{A}$ is an algebra, separates points, vanishes nowhere, and/or is self-adjoint?
5. Given a reasonably nice function $f : [-\pi, \pi] \to \mathbb{C}$, you should be able to compute its Fourier coefficients. If they decay sufficiently fast (so that you may apply Theorem 8.22), could you determine if f is equal to its Fourier series?
6. For a function $f \in R(\mathbb{T})$, you should be able to apply Parseval's theorem to compute (either side) of
$$\|f\|_2^2 = \sum_{n \in \mathbb{Z}} \left| \widehat{f}(n) \right|^2.$$

Things not to worry about:

I've tried to be fairly explicit about how deeply you need to know the results listed above and their proofs. Still, the course notes contain much more than we covered. While it might make sense to read the following section/results of the notes, you won't be responsible for anything in them (beyond what we did cover in lecture):

1. Section 7.3
2. Section 8.3 (though you should understand that everything here is simply rephrasing what we've already done (in terms of convergence, etc.) for 2π-periodic functions (i.e., functions on $\mathbb{T}$).
3. Proposition 8.24
4. Theorem 8.36
5. Definition 8.37
6. Subsection 8.5.3.