Math 338: Homework 8

Please complete the exercises below and write up your solutions consistent with the directions in the syllabus. Your solutions are due on Thursday, April 25th in the appropriate box outside my office door by 10AM. If you get stuck on any part of the homework, please come and see me. More importantly, have fun!

Exercise 1. Please do the following:

1. Please do Exercise 1 from Chapter 5 in Rudin.
2. Let $f : \mathbb{R} \to \mathbb{R}$ be α-Hölder continuous of order $\alpha > 1$. Prove that f is constant.

Exercise 2. Use the MVT or Taylor's theorem to prove the following statement:

Proposition 1. *Let $f : [a, b] \to \mathbb{R}$ be twice differentiable and $f''(x) > 0$ for all $x \in (a, b)$. then for any $x_0 \in (a, b)$,*

$$T(x) := f'(x_0)(x - x_0) + f(x_0) \leq f(x)$$

for all $x \in [a, b]$.

Also, give an interpretation of this statement.

Exercise 3. Please do the following.

1. Please do Exercise 26 from Chapter 5 in Rudin.
2. Please do Exercise 27 from Chapter 5 in Rudin.

Exercise 4. A region $\mathcal{R} \subseteq \mathbb{R}^d$ is said to be *uniformly quasi-convex* if there is a constant $M > 0$ such that, for any two points $\mathbf{x}, \mathbf{y} \in \mathcal{R}$, we can find a differentiable function $\mathbf{r} : [0, 1] \to \mathcal{R}$ with the property that $\mathbf{r}(0) = \mathbf{x}$, $\mathbf{r}(1) = \mathbf{y}$ and

$$\|D\mathbf{r}(t)\| \leq M\|\mathbf{x} - \mathbf{y}\|$$

for all $t \in [0, 1]$. Please do the following:

1. Show that the interval $[a, b]$ is uniformly quasi-convex in $\mathbb{R}$.
2. If $B \subseteq \mathbb{R}^d$ is convex, i.e., for any two points $\mathbf{x}, \mathbf{y} \in B$, $(1 - t)\mathbf{x} + t\mathbf{y} \in B$ whenever $0 \leq t \leq 1$, then B is also uniformly quasi-convex.
3. Show that the annulus,
$$A = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \middle| \ 1 \leq \sqrt{x^2 + y^2} \leq 2 \right\}$$
is uniformly quasi-convex.
4. If $\mathcal{R} \subseteq \mathbb{R}^d$ is uniformly quasi-convex and $f : \mathcal{R} \to \mathbb{R}$ is differentiable with bounded derivative vector/matrix, i.e., there is a constant K for which $\|Df(\mathbf{x})\| \leq K$ for all $\mathbf{x} \in \mathcal{R}$, then f is Lipshitz.