Math 338: Homework 6

Please complete the exercises below and write up your solutions consistent with the directions in the syllabus. Your solutions are due on Thursday, April 11th at 10:00AM in the appropriate box outside my office door. If you get stuck on any part of the homework, please come and see me. More importantly, have fun!

We begin the exercises this week dealing with continuous (or not) functions mapping from $\mathbb{R}$ to itself, i.e., real functions.

Exercise 1 (Continuity and some uniform continuity)**.** In this exercise, all functions map from $\mathbb{R}$ (with, perhaps, the domain being a proper subset) into $\mathbb{R}$. First,

1. (a) Proceding directly by definition (Definition 4.5), prove that the following function is continuous on its natural domain:
$$f(x) = \frac{1}{x}$$
   (b) Proceding directly by definition (Definition 4.5), prove that
$$g(x) = \begin{cases} x\sin\left(\frac{1}{x}\right) & x \neq 0 \\ 0 & x = 0 \end{cases}$$
   is continuous at 0.

   (c) Let $h : \mathbb{R} \to \mathbb{R}$ be defined by
$$h(x) = \begin{cases} x & x \in \mathbb{R} \setminus \mathbb{Q} \\ 0 & x \in \mathbb{Q} \end{cases}.$$
   Prove that h is continuous at 0 and discontinuous at all other points.

   Hint: For (b), you can use the fact that $|\sin(x)| \leq 1$ for all $x \in \mathbb{R}$. Admittedly, we haven't discussed the sine function yet in detail. Does your argument make use of any other (special) property of sine?

2. Consider the function $f : (0, 1] \to \mathbb{R}$ defined by $f(x) = 1/x$.
   (a) Prove that f is not uniformly continuous on its domain.
   (b) Prove that, for any $0 < r < 1$, f is uniformly continuous on $[r, 1]$.

Exercise 2. In this exercise, $f : [0, 1] \to \mathbb{R}$ is a continuous function (with $\mathrm{dom}(f) = [0, 1]$). Prove or refute the following statements.

1. If $f(p) > 0$ for some $p \in [0, 1]$, then there exists $\delta > 0$ for which $f(x) > 0$ for all $x \in [0, 1]$ such that $|x - p| < \delta$, i.e., f is strictly positive on an interval containing p.
2. If $f(p) = 0$ for some $p \in [0, 1]$ then there exists $\delta > 0$ for which $f(x) = 0$ for all $x \in [0, 1]$ such that $|x - p| < \delta$.
3. If $f(q) = 0$ for all rational numbers $q \in \mathbb{Q} \cap [0, 1]$, then $f(x) = 0$ for all $x \in [0, 1]$.
4. If $g : [0, 1] \to \mathbb{R}$ is another continuous function with $f(0) \leq g(0)$ and $f(1) \geq g(1)$, then there exists some $0 \leq c \leq 1$ such that $f(c) = g(c)$.

Let's now move on to metric spaces, in general, and continuous functions between. First, the following exercise is a nice generalization of the first item of Exercise 2. It turns out to be quite useful when studying integration, as we will see.

Exercise 3. This little exercise turns out to be pretty useful when proving things about the Riemann integral, as we shall see. Let (X, d) be a metric space and let f be a continuous function from X to $\mathbb{R}$. Prove the following: If $f(p) > 0$ for some $p \in X$, then there are positive constants M and r for which

$$M \leq f(x)$$

for all $x \in N_r(p)$ (and x in the domain of f).

Exercise 4. Let f be a continuous function from a metric space (X, d) into $\mathbb{R}$ (equipped with the usual metric) and assume that $\text{dom}(f) = X$.

1. Prove that the so-called zero set,
$$Z(f) = \{x \in X \mid f(x) = 0\}$$
is closed.

2. Let g be another continuous function from (X, d) into $\mathbb{R}$. Prove that
$$\{x \in X \mid f(x) < g(x)\}$$
is open and
$$\{x \in X \mid f(x) \leq g(x)\}$$
is closed.

Before we get the the next exercise, consider the following definition.

Definition 1. Let (X, d_X) and (Y, d_Y) be metric spaces and $f : X \to Y$. We say that f is Hölder continuous with parameter $\alpha > 0$ if there is some constant M for which

$$d_Y(f(p), f(q)) \leq M(d_X(p, q))^\alpha$$

for $p, q \in \text{dom}(f)$. In this case, we will call α the Hölder parameter for f and also say that f is α-Hölder continuous. In the special case that f is Hölder continuous with parameter $\alpha = 1$, f is also called Lipshitz.

Exercise 5. Let (X, d_X) and (Y, d_Y) be metric spaces. Please do the following:

1. Prove that Hölder continuous functions are uniformly continuous.

2. Show that constant functions are α-Hölder continuous for every $\alpha > 0$.

3. Write down what it means for a function $f : \mathbb{R}^d \to \mathbb{R}$ to be α-Hölder continuous. Note: Here I'm just asking you to rewrite the condition using the normal metrics on the domain and codomain of f. What about when $d = 1$?

4. Given $a = (a_1, a_2, \ldots, a_d) \in \mathbb{R}^d$ and $b \in \mathbb{R}$, define $f : \mathbb{R}^d \to \mathbb{R}$ by
$$f(x) = a \cdot x + b = a_1x_1 + a_2x_2 + \cdots a_dx_d + b$$
for $x = (x_1, x_2, \ldots, x_d) \in \mathbb{R}^d$. Prove that f is Lipshitz.

5. Given $0 < \alpha < 1$, find a non-constant α-Hölder continuous function on $X = \mathbb{R}$.

6. Challenge (you don't have to do this one): For $\alpha > 1$, can you find a non-constant α-Hölder continuous function on $X = [0, 1]$?

Exercise 6. Let (X, d) be a metric space and $A \subseteq X$. We define the distance between a point $x \in X$ and the set A to be

$$d(x, A) = \inf_{y \in A} d(x, y).$$

1. To get an understanding of $x \mapsto d(x, A)$, draw a picture of what it represents/measures (say with a shape A in $\mathbb{R}^2$) at various points.

2. Prove that the function $x \mapsto d(x, A)$ (which is a function from X to $\mathbb{R}$) is Lipshitz and is therefore continuous.

Before our next exercise, let's recall the following definition from Homework 3:

Definition 2. Let (X, d) be a metric space and let E and F be subsets of X.

1. We say that E and F are separated if $E \cap \overline{F} = \varnothing$ and $\overline{E} \cap F = \varnothing$.

2. We say that E and F are disconnected if there are open sets $\mathcal{O}_E$ and $\mathcal{O}_F$ for which $E \subseteq \mathcal{O}_E$, $F \subseteq \mathcal{O}_F$, and $\mathcal{O}_E \cap \mathcal{O}_F = \varnothing$.

In Exercise 4 of Homework 3, you proved the following statement: If E and F are disconnected, then they are separated. To jog your memory, here is a proof.

Proof. Given that E and F are disconnected, we may take disjoint open sets $\mathcal{O}_E$ and $\mathcal{O}_F$ for which $E \subseteq \mathcal{O}_E$ and $F \subseteq \mathcal{O}_F$. Set $C := X \setminus \mathcal{O}_E$ and observe that, because $\mathcal{O}_E$ and $\mathcal{O}_F$ are disjoint and $F \subseteq \mathcal{O}_F$, we have

$$F \subseteq \mathcal{O}_F \subseteq X \setminus \mathcal{O}_E = C.$$

Also, C is the complement of an open set and is therefore closed (Theorem 2.3). So, since C is a closed set containing F, Theorem 2.27 guarantees that

$$E \cap \overline{F} \subseteq E \cap C \subseteq \mathcal{O}_E \cap C = \varnothing$$

so $E \cap \overline{F} = \varnothing$. By a completely analogous (and symmetric) argument, we find that $\overline{E} \cap F = \varnothing$ and thus E and F are separated. $\square$

In fact, the converse to the above is also true, i.e., we have:

Theorem 3. *Let (X, d) be a metric space and E and F be subsets of X. Then E and F are separated if and only if E and F are disconnected.*

Exercise 7. Complete the proof of the theorem above by showing that if E and F are separated, then they are disconnected.

Hint: Consider the set $\{x \in X \mid d(x, E) < d(x, F)\}$ and its symmetric analogue.

Exercise 8. Let X and Y be metric spaces with metrics d_X and d_Y, respectively. Let $f : X \to Y$ be uniformly continuous on X. Please do the following:

1. Prove that, if $\{p_n\}$ is Cauchy sequence in X, then $\{f(p_n)\}$ is a Cauchy sequence in Y.

2. If X has the property that every bounded infinite collection of points has a limit point (called a Bolzano-Weierstrass space), prove the following: If $S \subseteq X$ is a bounded set, then f is bounded on S, i.e., there is $q \in Y$ and $M > 0$ for which
$$f(S) \subseteq N_M(q) = \{y \in Y \mid d_Y(y, q) < M\}.$$

3. Use the result above to give another proof that $1/x$ cannot be uniformly continuous on $(0, 1]$.