Math 338: Homework 3 –Leap Homework

Please complete the following exercises below and write up your solutions consistent with the directions in the syllabus. Your solutions are due on Thursday, February 29th at 10:00AM in the appropriate box outside my office door. If you get stuck on any part of the homework, please come and see me. More importantly, have fun!

Exercise 1. Let (X, d) be a metric space and $E \subseteq X$. The **boundary of** E is defined to be the set

$$\partial E = \overline{E} \cap \overline{E^c}.$$

1. Prove that E is open if and only if $E \cap \partial E = \varnothing$.
2. Prove that E is closed if and only if $\partial E \subseteq E$.

Exercise 2 (Going back through the proof of Theorem 2.38)**.** Let $\{I_n\}$ be a sequence of intervals in $\mathbb{R}$ of the form $I_n = [a_n, b_n]$ for $a_n \leq b_n$. Suppose that $\{I_n\}$ are nested in the sense that $I_n \subseteq I_m$ whenever $n \geq m$. Prove that $\inf\{b_n\}$ exists and belongs to the intersection

$$\bigcap_{n=1}^{\infty} I_n.$$

Exercise 3. Let (X, d) be a metric space and let $E \subseteq X$. We say that E **is bounded** if there is some $x \in X$ and $r > 0$ for which $E \subseteq N_r(x)$. In the case that E is non-empty, we can also define the **diameter** of E to be the extended real number

$$\operatorname{diam}(E) = \sup \{d(x, y) \mid x, y \in E\}.$$

1. In the case that $X = \mathbb{R}$ with its usual metric and $a < b$ are real numbers, show that $E = (a, b)$ is bounded and compute $\operatorname{diam}(E)$.
2. We return to the general setting in which E is some non-empty set of a metric space (X, d). Prove that $\operatorname{diam}(E) < \infty$ whenever E is bounded.
3. Also, prove the converse: E is bounded whenever $\operatorname{diam}(E) < \infty$.
4. Prove the following theorem:

   Theorem A (Cantor's Intersection Theorem)**.** *Let (X, d) be a metric space and let $\{K_n\}$ be an infinite collection of non-empty compact sets which are nested in the sense that $K_m \subseteq K_n$ whenever $m \geq n$. If, for each $\epsilon > 0$, there are only finitely many K_n for which $\operatorname{diam}(K_n) \geq \epsilon$, then*

   $$\bigcap_n K_n$$

   contains exactly one point.

Hint: If the intersection contained two distinct points, could this be an issue for the diameter hypothesis?

Exercise 4. Let (X, d) be a metric space and E and F be sets. The sets E and F are said to be **disconnected** if there are disjoint open sets $\mathcal{O}_E$ and $\mathcal{O}_F$ with $E \subseteq \mathcal{O}_E$ and $F \subseteq \mathcal{O}_F$. Prove the following statement: If E and F are disconnected, then E and F are separated. **Note:** It turns out that the converse is true as well. We'll get to it later.

Exercise 5. Please do Baby Rudin's Exercise 19 of Chapter 2. An extra hint for (d): Suppose that X contains two distinct points $p \neq q$ and define $f(x) = d(x, p)$ for $x \in X$. If X were countable, would it be possible to f to map onto the interval $[f(p), f(q)] = [0, f(q)]$?

Exercise 6. Please do Baby Rudin's Exercise 30 of Chapter 2. **Hint:** To do this problem, it's necessary to really think about/digest the ideas behind the proof of Theorem 2.43. Personally, I don't find Rudin's proof extremely clear. Abbott's proof (Theorem 3.4.3 in Abbott) is much more clear; however, it should be noted that Abbott only works in the real line and there are a couple of small typographical errors in his proof.

Some extra exercises

The exercises below have to do with "metric subspaces". I'm not going to collection solutions to these, but they are good practice.

Definition B. Let (X, d) be a metric space and Y be a subset of X. We define d_Y by

$$d_Y(p, q) = d(p, q)$$

whenever $p, y \in Y$. This means that d_Y is the restriction of the metric d to the subset Y.

Exercise 7. As you show below, given a metric space (X, d) and $Y \subseteq X$, (Y, d_Y) is itself a metric space. We shall refer to (Y, d_Y) (or simply Y by an abuse of notation) as a *metric subspace* (or simply *subspace*) of (X, d). In this context, (X, d) is called the ambient space. To distinguish between neighborhoods, we shall write

$$N_{Y,r}(p) = \{q \in Y \mid d_Y(p, q) < r\}$$

to denote the neighborhood in Y of radius $r > 0$ centered at $p \in Y$; $N_r(p)$ will still refer to a neighborhood in the ambient space.

1. Prove that d_Y is a metric on Y and hence (Y, d_Y) is a metric space.
2. In what follows, we take $X = \mathbb{R}$ to be equipped with the usual metric, $d(p, q) = |p - q|$, and consider various subsets/spaces Y which we take equipped with the subspace metric d_Y and subsets $E \subseteq Y$. For each, determine if E is open in the metric spaces (Y, d_Y) and (X, d). Prove your assertions.
   (a) $Y = \mathbb{N}$ and, for any fixed $n \in \mathbb{N}$, the "singleton" $E = \{n\}$.
   (b) $Y = \mathbb{Q}$ and, for any fixed $q \in \mathbb{Q}$, the singleton $E = \{q\}$.
   (c) $Y = (a, b]$ and $E = (c, b]$ where $a \leq c < b$.
   (d) $Y = (a, b]$ and $E = (a, d]$ where $a < d \leq b$.

We take the following definition from Rudin:

Definition C. Let (X, d) be a metric space and Y be a subset of X. A set $E \subseteq Y$ is said to be open relative to Y if, for every $p \in E$, there is a positive real number $r > 0$ for which $q \in E$ whenever $d(p, q) < r$ and $q \in Y$.

Exercise 8. Let (X, d) be a metric space and Y be a subset of X.

1. Show that a set $E \subseteq Y$ is open relative to Y if and only if it is open in the metric space (Y, d_Y).

In view of Theorem 2.30 and the previous item, it follows that E is an open set in (Y, d_Y) if and only if $E = Y \cap G$ for some open set G in X. In other words, we have

Theorem D. *Let (X, d) be a metric space, $Y \subseteq X$, and let (Y, d_Y) be the associated metric subspace. Then, for a set $E \subseteq Y$, the following are equivalent:*

a. E is open relative to Y.

b. E is open in the metric space (Y, d_Y)..

c. $E = Y \cap G$ where G is an open set in (X, d).

Now that you are armed with the theorem above, do the following to complete the exercise.

2. For each set E in the previous exercise that you found to be open in (Y, d_Y), find an open set $G \subseteq \mathbb{R}$ for which $E = Y \cap G$.
3. Let's return to the general assumption that (X, d) is a metric space and $Y \subseteq X$. Regarding the theorem above, is the same assertion true for closed sets? In other words, is it true that $F \subseteq Y$ is closed in (Y, d_Y) if and only if $F = Y \cap H$ where H is closed in (X, d)? Prove your assertion or find a counterexample.
4. Onto compact sets: Is it true that $J \subseteq Y$ is compact in (Y, d_Y) if and only if $J = Y \cap K$ for some compact set in K in (X, d)? Prove your assertion or find a counterexample.