Math 338: Homework 1

Please complete the following exercises below and write up your solutions consistent with the directions in the syllabus[1]. Your solutions are due on Thursday, February 15th at 10:00AM in the appropriate box outside my office door. If you get stuck on any part of the homework, please come and see me. More importantly, have fun!

Exercise 1. In this exercise, $\mathbb{Q}$ denotes the rational number field with its usual ordering. Consider the subset

$$A = \left\{ \frac{1}{n} : n = 1, 2, \ldots \right\}$$

of $\mathbb{Q}$.

1. Prove that A is bounded above and below.
2. Prove that 1 is the supremum of A.
3. Prove that 0 is the infimum of A.
4. Since 0 and 1 are both rational numbers, why doesn't this example show that $\mathbb{Q}$ has the least upper bound property?

Exercise 2. When mathematicians use the word "the" to refer to an object, there is an implicit assumption that it is unique, i.e., it is the only one. The indefinite articles "a" or "an" allow for the possibility that there can be more than one of such object.

1. By using examples, explain why we would say "an upper bound" or "a lower bound" instead of using the definite article "the".
2. If you notice, when we introduced the notions of supremum and infimum, we used the definite article "the". To explain this, prove the following statement:

   Let S be an ordered set and let E be a non-empty subset of S. Prove that E can have at most one supremum.

   Hint: Assume that α_1 and α_2 both satisfy the definition of supremum of E. Prove that $\alpha_1 = \alpha_2$.

Exercise 3. Let F be an ordered field and E a non-empty subset of F.

1. If α is a lower bounded of E and β is an upper bound of E, prove that $\alpha \leq \beta$. Did you use anything about F being a field?
2. Assume that E is a bounded below and that $\inf E$ exists (as an element of F, minimally). Define

   $$-E = \{-x \in F : x \in E\}.$$

   Prove that

   $$\sup(-E) = -\inf E.$$

3. Assume that, E is a subset of the strictly positive elements of F, i.e, for all $x \in E$, $x > 0$, and define

   $$1/E := \{1/x : x \in E\}.$$

   If $\alpha = \sup E$, prove that $1/\alpha = \inf(1/E)$.

[1] Now is an excellent time, if you haven't already, to read the syllabus carefully.

In the course of your proofs, indicate which propositions (if any) you used in the "Fields" section of Baby Rudin's Chapter 1. Be precise please.

Exercise 4. Please do Baby Rudin's Exercise 6 of Chapter 1.

Exercise 5. Please do Baby Rudin's Exercise 7 of Chapter 1.

Exercise 6. Please do Baby Rudin's Exercise 11 of Chapter 1.

Exercise 7. Please do Baby Rudin's Exercise 12 of Chapter 1.

Exercise 8. Please do Baby Rudin's Exercise 13 of Chapter 1.