# MATH 253: PROJECT II

## EIGENVECTORS AND EIGENVALUES

### 1. Introduction

Due to our shortened and chaotic semester, we were not able to cover the material of Chapter 5 as we had originally planned. The purpose of this project is to expose you to the basics of eigenvectors and eigenvalues, an essential topic in linear algebra, in a way that I hope will be useful to you as you move on from this course.

The remainder of this document covers eigenvectors, eigenvalues, and eigenspaces. It is complete with definitions, results, examples, and exercises. Your job for this project is to master this document and complete all exercises herein. To this end, you will read this document thoroughly and do the exercises. Then, please write up your exercise solutions in a well-polished publication-quality document (PDF form, typed or hand-written). You are permitted to discuss this project with your classmates[1], however, when it comes time to write up your solutions, you are required to do so independently. You are **not** permitted to discuss this project with anyone outside of this course (MA253, Section A), i.e., you may not seek help on internet forums nor from anyone not associated with this class – doing so is a matter of academic dishonesty[2]. Your solutions should communicate your individual process and understanding of the material and they must be in your own words and your own voice. **What you turn in must be your own**. I expect your solutions to be written out correctly and presented in good mathematical prose. Your grade will depend on the correctness of your solutions and the quality of your writing. This means that your writing should follow a coherent logical structure which makes use of complete sentences and follows standard rules of grammar. Please do not submit solutions containing incoherent and unstructured calculations.

#### 1.1. Due Date, Getting Help, Etc.

(1) If you have questions as you're working, feel free to email me and I'll happily help via email (or we can set up a zoom session).
(2) If you discuss this project with a classmate, make sure to list that classmate's name in the body of your project submission email.
(3) In the course of a solution, if you make use of a result from the textbook (Chapters 1-4), please make sure to cite it appropriately and give a page number.
(4) Please do not use material beyond this document and Chapters 1-4 of our textbook. I assure you that nothing more is needed for a perfect solution.
(5) This project is officially due on May 8th, but I will accept it for full credit as late as Tuesday, May 12th by 11:59PM (in your respective time zone).
(6) You solutions should be in PDF form and emailed to me with the subject line (adjusted appropriately):

    [MA253] Project 2; Doe, Jane

    Note: There is no space between MA and 253.

[1] If you do, please make a note in the body of your submission email and list those classmates with whom you discussed the project.

[2] I encourage you to see Colby's web-site on academic integrity (here) for more information and helpful resources of good practices of academic integrity.

(7) If you would like to typeset your solutions in LaTeX, please email me and I can help get you started.
(8) If you have questions about anything (the subject matter, feel like you've found a typo, the write-up structure, etc.), please email me! I'm here to help.

## 2. Eigenvectors and Eigenvalues

**Definition 2.1.** *Let $V$ be a real vector space and $T : V \to V$ a linear transformation from $V$ to itself. A non-zero vector $\mathbf{v} \in V$ is said to be an eigenvector for $T$ if*

$$T(\mathbf{v}) = \lambda \mathbf{v}$$

*for some scalar $\lambda$. In this case, $\lambda$ is said to be an eigenvalue of $T$ and, further, $\mathbf{v}$ is said to be an eigenvector of $T$ with corresponding eigenvalue $\lambda$.*

Essentially, the eigenvectors of a linear transformation $T$ are special vectors that are mapped to scalar multiples of themselves by $T$. The notion at hand is entirely dependent on the nature of the linear transformation $T$; unless $T$ is very special (e.g., the identity transformation), it's "rare" for a given vector to be an eigenvector. The notion of eigenvectors is extremely important for many topics in linear algebra and its applications.

**Example 1**

Consider the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ with standard matrix representation

$$A = \begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix},$$

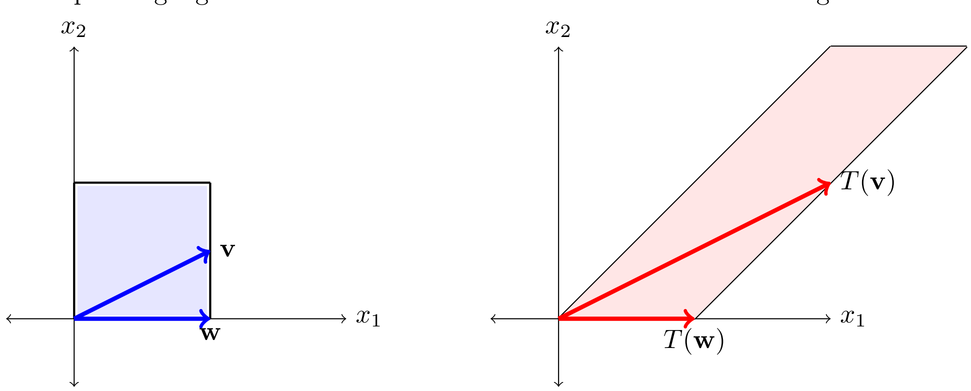
i.e.,

$$T(\mathbf{x}) = A\mathbf{x} = \begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 + 2x_2 \\ 2x_2 \end{pmatrix}$$

for $\mathbf{x} = (x_1, x_2)^\intercal \in \mathbb{R}^2$. It should be fairly straightforward to see that $T$ is the composition of dilation and shear transformations. Observe that, for $\mathbf{v} = (2,1)^\intercal \in \mathbb{R}^2$,

$$T(\mathbf{v}) = \begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \end{pmatrix} = 2 \begin{pmatrix} 2 \\ 1 \end{pmatrix} = 2\mathbf{v}.$$

Thus, $\mathbf{v} = (2,1)^\intercal$ is an eigenvector for $T$ with corresponding eigenvalue $\lambda = 2$. On your own, you should check that $\mathbf{w} = (2,0)^\intercal$ is also an eigenvector for $T$ with corresponding eigenvalue $\lambda = 1$. These vectors are illustrated in the figure below.




In studying the figure above, you should think about why these specific vectors $\mathbf{v} = (2,1)^\intercal$ and $\mathbf{w} = (2,0)^\intercal$ are mapped to scalar multiples of themselves by $T$. Do you see any other vectors that are also taken to scalar multiples of themselves by $T$?

**Exercise 1**

Consider the set of $n$th-degree polynomials $\mathcal{P}_n = \{p(t) = a_0 + a_1 t + \cdots + a_n t^n\}$. As we've seen $\mathcal{P}_n$ is a vector space. Consider the map $T : \mathcal{P}_n \to \mathcal{P}_n$ defined by

$$T[p](t) = t \cdot p'(t) = t\left(0 + a_1 + 2a_2 t + \cdots n a_n t^{n-1}\right) = a_1 t + 2a_2 t^2 + \cdots + n a_n t^n$$

for each polynomial $p(t) = a_0 + a_1 t + a_2 t^2 + \cdots a_n t^n \in \mathcal{P}_n$.

(1) Show that $T$ is linear.
(2) Show that the "constant polynomials" are all eigenvectors of $T$. That is, for each non-zero constant $a_0$, show that $p_0(t) = a_0 + 0t + 0t^2 + \cdots 0t^n \in \mathcal{P}_n$ is an eigenvector of $T$. What is its corresponding eigenvalue?
(3) Show that the "$n$th order monomials" are all eigenvectors of $T$. That is, for each non-zero constant $a_n$, show that $p_n(t) = 0 + 0t + \cdots 0t^{n-1} + a_n t^n$ is an eigenvector of $T$. What is its corresponding eigenvalue?
(4) Can you think of other eigenvectors? If so, write down any you can find and give their corresponding eigenvalues.

In looking back to Example 2 and Exercise 2, you might have observed that any constant multiple of an eigenvector is also an eigenvector (with the same eigenvalue). For example, the vector $6\mathbf{v} = (12, 5)^\intercal$ is an eigenvector of the transformation $T$ (with Eigenvalue $\lambda = 2$) in Example 2. To summarize this observation, we have the following proposition.

**Proposition 2.2.** *Let $V$ be a vector space and $T : V \to V$ be a linear transformation. If $\mathbf{v}$ is an eigenvector for $T$ with corresponding eigenvalue $\lambda$ then, for any non-zero constant $\alpha$, $\alpha\mathbf{v}$ is also an eigenvector for $T$ with eigenvalue $\lambda$.*

*Proof.* For $\alpha \neq 0$, $\alpha\mathbf{v}$ is necessarily a non-zero vector and

$$T(\alpha\mathbf{v}) = \alpha T(\mathbf{v}) = \alpha\lambda\mathbf{v} = \lambda\alpha\mathbf{v} = \lambda(\alpha\mathbf{v})$$

and hence the vector $\alpha\mathbf{v}$ is an eigenvector with eigenvalue $\lambda$. □

In view of the proposition above, we observe immediately that there are always lots (an infinite number) of eigenvectors corresponding to a single eigenvalue. Namely, any scalar multiple of an eigenvector is also an eigenvector with the same eigenvalue. The following example shows that this idea goes beyond scalar multiples.

**Example 2**

Consider the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ with standard matrix representation

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

and vectors $\mathbf{v} = (1, 0, 0)^\intercal$ and $\mathbf{w} = (0, 0, 1)^\intercal$. We have

$$T(\mathbf{v}) = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = 1 \cdot \mathbf{v}$$

and

$$T(\mathbf{w}) = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = 1 \cdot \mathbf{w}$$

and therefore $\mathbf{v}$ and $\mathbf{w}$ are both eigenvectors of $T$ and, in fact, both have corresponding eigenvalue $\lambda = 1$. In Proposition 2.2, we know that all scalar multiples of eigenvectors are also eigenvectors (of the same eigenvalue). This example shows that not all eigenvectors corresponding to a single eigenvalue are scalar multiples of one another. In fact, for any scalars $\alpha, \beta \in \mathbb{R}$, consider the linear combination

$$\mathbf{u} = \alpha\mathbf{v} + \beta\mathbf{w} = \alpha \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \alpha \\ 0 \\ \beta \end{pmatrix}$$

and observe that

$$T(\mathbf{u}) = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} \alpha \\ 0 \\ \beta \end{pmatrix} = \begin{pmatrix} \alpha \\ 0 \\ \beta \end{pmatrix} = 1 \cdot \mathbf{u}.$$

Correspondingly, any non-zero linear combination of the eigenvectors $\mathbf{v}$ and $\mathbf{w}$ with corresponding eigenvalue $\lambda = 1$ is another eigenvector with eigenvalue $\lambda = 1$.

The observations made in the preceding example lead us to consider the following general notion. Let $V$ be a real vector space and $T : V \to V$ a linear transformation. If $\lambda$ is an eigenvalue of $T$ (that is, there is an eigenvector $\mathbf{v}$ for $T$ with this corresponding eigenvalue $\lambda$), we define the *eigenspace of $T$ corresponding to $\lambda$* to be the set

$$E_\lambda = \{\mathbf{v} \in V : T(\mathbf{v}) = \lambda\mathbf{v}\}.$$

The use of the terminology "eigenspace" (say, over "eigenset") is justified by the following exercise.

**Exercise 2**

Given $V$, $T$, $\lambda$, and $E_\lambda$ be as above.

(1) In a single sentence, explain why each eigenvector $\mathbf{v}$ of $T$ with corresponding eigenvalue $\lambda$ is a member of the set $E_\lambda$.
(2) Are there any vectors in $E_\lambda$ which are not eigenvectors of $T$? Justify your answer. Hint: Remember eigenvectors are required, by definition, to be non-zero.
(3) Prove that $E_\lambda$ is a subspace of $V$.
(4) If $\lambda$ were not an eigenvalue of $T$, would the set $E_\lambda$ still be a subspace of $V$? Justify your answer.

Given a linear transformation $T : V \to V$ from a real vector space $V$ to itself, how does one find the eigenvectors and eigenvalues of $T$? One key to answering this question is to recognize that the spaces $E_\lambda$ are kernels of related linear transformations. To this end, given a scalar $\lambda$, consider the equation

$$T(\mathbf{v}) = \lambda\mathbf{v},$$

which may or may not have non-zero solutions $\mathbf{v} \in V$. This equation can be equivalently rewritten as

$$(T - \lambda I)(\mathbf{v}) = T(\mathbf{v}) - \lambda\mathbf{v} = \mathbf{0}$$

where $I : V \to V$ denotes the identity transformation. Regardless of whether or not $\lambda$ is an eigenvalue, we see immediately that

$$E_\lambda = \ker(T - \lambda I).$$

Of course, the result of Exercise 2 is (re)confirmed since the kernel of the linear operator $(T-\lambda I):V\to V$ is always a subspace of $V$. In view of this observation, the question at hand becomes whether or not this kernel is trivial (i.e., containing only the zero vector).

**Proposition 2.3.** *Let $V$ be a real vector space and $T:V\to V$ a linear transformation. Given any scalar $\lambda\in\mathbb{R}$,*

$$E_\lambda=\ker(T-\lambda I)$$

*is a subspace of $V$. Furthermore, the following are equivalent:*
*(a) $\lambda$ is an eigenvalue of $T$.*
*(b) The subspace $E_\lambda$ is non-trivial (contains more than the zero vector).*
*(c) The linear transformation $(T-\lambda I):V\to V$ is not injective, i.e., it is not one-to-one.*

*Proof.* Since $T-\lambda I$ is a linear transformation from $V$ to itself, its kernel is always a subspace of $V$. To prove that conditions $(a)$, $(b)$ and $(c)$ are equivalent, we will show that $(a)$ implies $(b)$, then $(b)$ implies $(c)$, and then $(c)$ implies $(a)$. This is often written in the shorthand "$(a)\implies(b)\implies(c)\implies(a)$".

$(a)\implies(b)$ Let $\lambda$ be an eigenvalue of $T$ and let $\mathbf{v}$ be an eigenvector of $T$ with eigenvalue $\lambda$ and note that, by definition, $\mathbf{v}\neq\mathbf{0}$. Consequently $\mathbf{v}$ is a non-zero vector for which $(T-\lambda I)(\mathbf{v})=T(v)-\lambda\mathbf{v}=0$ showing that $E_\lambda$ contains the non-zero vector $\mathbf{v}$.

$(b)\implies(c)$ Suppose that $E_\lambda=\ker(T-\lambda I)$ contains a non-zero vector. In view of the characterization of injectivity (one-to-oneness), $T-\lambda I$ is not injective (not one-to-on).

$(c)\implies(a)$ Suppose that $(T-\lambda I)$ is not injective. Then, in view of the characterization of injectivity in terms of the kernel, there must be a non-zero vector $\mathbf{v}$ for which $(T-\lambda I)(\mathbf{v})=\mathbf{0}$. Consequently, $\mathbf{v}$ is a non-zero vector for which $T(\mathbf{v})=\lambda\mathbf{v}$ whence $\mathbf{v}$ is an eigenvector with corresponding eigenvalue $\lambda$. ☐

**Example 3**

Consider the vector space $V=\mathbb{R}^{\mathbb{N}}$ of real sequences. As we saw in Chapter 4, each vector $\mathbf{a}\in V$ is a sequence of the form

$$\mathbf{a}=\{a_k\}_{k\in\mathbb{N}}=(a_0,a_1,a_2,\dots)$$

where, for each $k=0,1,2,\dots$, the entry $a_k$ is a real number. We recall that addition and scalar multiplication is defined by the following formulae: For $\mathbf{a}=(a_0,a_1,a_2,\dots)$ and $\mathbf{b}=(b_0,b_1,b_2,\dots)$ and a scalar $\kappa\in\mathbb{R}$,

$$\mathbf{a}+\mathbf{b}=(a_0+b_0,a_1+b_1,a_2+b_2,\dots)$$

and

$$\kappa\mathbf{a}=(\kappa a_0,\kappa a_1,\kappa a_2,\dots).$$

Consider the so-called *left shift transformation* $L:V\to V$ given by

$$L(\mathbf{a})=(a_1,a_2,a_3,\dots)$$

for each sequence $\mathbf{a}=(a_0,a_1,a_2,\dots)\in V$. For example, given the sequence $(0,2,4,6,8,\dots)\in V$, we have

$$L((0,2,4,6,8,\dots,))=(2,4,6,8,10,\dots).$$

For each scalar $\lambda\in\mathbb{R}$, to understand the subspace $E_\lambda\subseteq V$, we consider the transformation $L-\lambda I$ where $I$ is the identity transformation on $V$. For each

$\mathbf{a} = (a_0, a_1, a_2, \dots) \in V$, we have

$$\begin{aligned} (L - \lambda I)(\mathbf{a}) &= L(\mathbf{a}) - \lambda I(\mathbf{a}) \\ &= (a_1, a_2, a_3, \dots) - \lambda(a_0, a_1, a_2, \dots) \\ &= (a_1 - \lambda a_0, a_2 - \lambda a_1, a_3 - \lambda a_2, \dots). \end{aligned}$$

Consequently, each element $\mathbf{a} \in \ker(L - \lambda I) = E_\lambda$ satisfies

$$(0, 0, 0, 0, \dots) = \mathbf{0} = (L - \lambda I)(\mathbf{a}) = (a_1 - \lambda a_0, a_2 - \lambda a_1, a_3 - \lambda a_2, \dots).$$

In other words, $\mathbf{a} = (a_0, a_2, a_3, \dots) \in \ker(L - \lambda I) = E_\lambda$ if and only if

$$a_1 = \lambda a_0, \;\; a_2 = \lambda a_1, \;\; a_3 = \lambda a_2, \dots$$

that is,

$$a_{k+1} = \lambda a_k$$

for all $k = 0, 1, 2, \dots$. Using this relation inductively, it's not hard to see that $\mathbf{a} = (a_0, a_1, a_2, \dots) \in \ker(L - \lambda I) = E_\lambda$ if and only if

$$a_k = \lambda^k a_0$$

for all $k = 0, 1, 2, \dots$ and this means that the sequence $\mathbf{a}$ can be written in the form

$$\mathbf{a} = (a_0, a_0\lambda, a_0\lambda^2, a_0\lambda^3, \dots) = a_0(1, \lambda, \lambda^2, \lambda^3, \dots).$$

You might remember from calculus (sequence and series) that these are so-called "geometric sequences" with common ratio $\lambda$. We have shown that, $\lambda \in \mathbb{R}$,

$$E_\lambda = \ker(L - \lambda I) = \{\mathbf{a} = a_0(1, \lambda, \lambda^2, \lambda^3, \dots) : a_0 \in \mathbb{R}\}.$$

**Exercise 3**

Let $V = \mathbb{R}^{\mathbb{N}}$ and $L$ be the left shift transformation in the preceding example. Please show the following[a]:

(1) $L$ is surjective (onto).
(2) $L$ is not injective (one-to-one).
(3) Each $\lambda \in \mathbb{R}$ is an eigenvalue of $L$.
(4) $E_\lambda$, the eigenspace of $L$ associated to $\lambda$, is a one-dimensional subspace of $V$.

Now, consider the *right shift transformation* $R : V \to V$ defined by

$$R(\mathbf{a}) = (0, a_0, a_1, a_2, \dots)$$

for each $\mathbf{a} = (a_0, a_1, a_2, \dots) \in V$. Show the following:

(3) $R : V \to V$ is linear.
(4) $R$ is injective (one-to-one).
(5) $R$ is not surjective (onto).
(6) $R$ has no eigenvalues, i.e., $E_\lambda$ is a trivial subspace for each $\lambda \in \mathbb{R}$.

Finally, to demonstrate some more interesting behaviors of the shift operators, do the following:

(7) Show that $L \circ R$ is the identity transformation.
(8) Show that $R \circ L$ is not the identity transformation.

[a] You may make use of the computations above and the results of Propostion 2.3

The above example and accompanying exercise might seem somewhat bizarre even though the shift transformations[3] are very simple. This type of strange behavior can only happen, as it turns out, when $V$ is infinite dimensional. If you're curious about these things, there is a vast literature on infinite-dimensional vector spaces (and feel free to talk to me

[3] The shift transformations make an interesting appearance in quantum mechanics.

about them if you are). Let's, however, turn our focus momentarily to finite-dimensional vector spaces (like $\mathbb{R}^n$).

We recall from Chapter 4 that, in the special case that a real vector space $V$ is finite-dimensional, linear transformations from $V$ to itself are injective (one-to-one) if and only if they are invertible. In view of this fact and the preceding proposition, we state the following useful corollary to Proposition 2.3.

**Corollary 2.4.** *Let $V$ be a finite-dimensional real vector space and $T : V \to V$ be a linear transformation. Then $\lambda$ is an eigenvalue of $T$ if and only if $(T - \lambda I)$ is singular (not invertible).*

Let's see the preceding proposition and corollary in action.

**Example 4**

Consider (again) the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ with standard matrix representation

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}.$$

Let's determine the eigenvalues of $T$ and their corresponding eigenspaces. In view of the preceding corollary, $\lambda$ is an eigenvalue of $T$ if and only if $(T - \lambda I)$ is singular. Of course, the operator $(T - \lambda I)$ is singular if and only if its (standard) matrix representation

$$(A - \lambda I_3) = \left( \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right) = \begin{pmatrix} 1-\lambda & 1 & 0 \\ 0 & 1-\lambda & 0 \\ 0 & 1 & 1-\lambda \end{pmatrix}$$

is not invertible (equivalently, has zero determinant). We therefore compute

$$\begin{aligned} 0 &= \det(A - \lambda I_3) \\ &= (1-\lambda)\det\left(\begin{pmatrix} 1-\lambda & 0 \\ 1 & 1-\lambda \end{pmatrix}\right) + (-1)(1)\det\left(\begin{pmatrix} 0 & 0 \\ 0 & 1-\lambda \end{pmatrix}\right) \\ &\qquad + (0)\det\begin{pmatrix} 0 & 1-\lambda \\ 0 & 1 \end{pmatrix} \\ &= (1-\lambda)(1-\lambda)^2 + (-1)(0) + (0)(0) \\ &= (1-\lambda)^3. \end{aligned}$$

Consequently, $\lambda$ is an eigenvalue of $T$ if and only if $(1-\lambda)^3 = 0$ and this happens if and only if $\lambda = 1$. Consequently, the only eigenvalue of $T$ is $\lambda = 1$. For this eigenvalue, let's determine the eigenspace

$$E_1 = \ker(T - 1I) = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3 : \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\}.$$

We see easily that $(x_1, x_2, x_3)^\intercal \in E_1$ if and only if $x_2 = 0$. Therefore,

$$E_1 = \left\{ \begin{pmatrix} x_1 \\ 0 \\ x_3 \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} : x_1, x_3 \in \mathbb{R} \right\} = \operatorname{Span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

Since $(1,0,0)^\intercal$ and $(0,0,1)^\intercal$ are linearly independent, they form a basis for $E_1$ and therefore we see that the eigenspace $E_1$ is two-dimensional.

**Exercise 4**

Consider the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ with standard matrix representation

$$A = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}.$$

Do the following:

(1) Determine whether or not $T$ is invertible.
(2) Find the eigenvalues of $T$.
(3) For each eigenvalue $\lambda$, characterize (as the example does above) the eigenspace $E_\lambda$. In particular, find a basis for $E_\lambda$ and state the dimension of $E_\lambda$.

Now, consider the linear transformation $S : \mathbb{R}^3 \to \mathbb{R}^3$ with standard matrix representation

$$B = \begin{pmatrix} 2 & 1 & -2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Do the following:

(4) Determine whether or not $S$ is invertible.
(5) Find all of the eigenvalues of $S$.
(6) For each eigenvalue $\lambda$ of $S$, characterize (as the example does above) the eigenspace $E_\lambda$. In particular, find a basis for $E_\lambda$ and state the dimension of $E_\lambda$.

In studying the above examples and exercises, injectivity and invertibility is related to (in some way) whether or not 0 is an eigenvalue of the given linear transformation. In this direction, consider the following exercise.

**Exercise 5**

Let $V$ be a real vector space and $T : V \to V$ a linear transformation. Observe that, in view of Proposition 2.3,

$$E_0 = \ker(T - 0I) = \ker(T).$$

By appealing to the results (and exercises) above, please do the following:

(1) In $\leq 2$ sentences, argue that $T$ is injective (one-to-one) if and only if 0 is not an eigenvalue of $T$.
(2) In $\leq 2$ sentences, argue that, when $V$ is finite-dimensional, $T$ is invertible if and only if 0 is not an eigenvalue of $T$.
(3) Consider now the left shift operator $L : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}^{\mathbb{N}}$ studied above. Give a single sentence comparing Item 1 (or 2) with your conclusions concerning whether or not $L$ was injective (or invertible) and whether or not 0 was an eigenvalue of $L$. For example, you could say: "In Exercise $x$, we showed that $L$ was (or was not) injective and also that 0 was (or was not) an eigenvalue of $L$, consistent with Item 1 (or Item 2) above."
(4) Do the same thing for the right shift operator, $R$.
(5) Do the same thing for the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ from the preceding exercise.
(6) Do the same thing for the linear transformation $S : \mathbb{R}^3 \to \mathbb{R}^3$ from the preceding exercise.

**Exercise 6**

Let $V$ be a real vector space and suppose that $T$ and $S$ are linear transformations from $V$ to itself. We say that $T$ and $S$ are *similar* if there is an invertible transformation $M : V \to V$ for which

$$M \circ T = S \circ M.$$

In what follows, we shall assume that $T$ and $S$ are similar with associated invertible transformation $M$.

(1) Show that

$$T = M^{-1} \circ S \circ M \qquad \text{and} \qquad S = M \circ T \circ M^{-1}$$

where $M^{-1}$ is the inverse of $M$.

(2) Show that $T$ and $S$ have the same eigenvalues, i.e., show that $\lambda$ is an eigenvalue of $T$ if and only if it is an eigenvalue of $T$.

(3) For each $\lambda \in \mathbb{R}$, define

$$E_{\lambda,T} = \ker(T - \lambda I)$$

and

$$E_{\lambda,S} = \ker(S - \lambda I)$$

to be the eigenspaces of $T$ and $S$ respectively. Show that

$$E_{\lambda,T} = \{v \in V : M(v) \in E_{\lambda,S}\}$$

and

$$E_{\lambda,S} = \{v \in V : M^{-1}(v) \in E_{\lambda,T}\}.$$

**Exercise 7: A glance at the spectral calculus**

Let $V$ be a real vector space and $T : V \to V$ be a linear transformation. For each integer $k \geq 1$, we define the linear transformation $T^k : V \to V$ by

$$T^k = \underbrace{T \circ T \circ T \circ \cdots \circ T}_{k \text{ times}},$$

i.e., for each vector $\mathbf{v} \in V$,

$$T^k(\mathbf{v}) = \underbrace{T(T(T \cdots (T}_{k}(\mathbf{v})) \cdots )).$$

Please do the following[a]:

(1) Show (or find a counterexample to) the following statement): If $\mathbf{v}$ is an eigenvector of $T$ with eigenvalue $\lambda$, then $\mathbf{v}$ is an eigenvector of $T^k$ with eigenvalue $\lambda^k$.

(2) Show (or find a counterexample to) the following statement: If $\mathbf{v}$ is an eigenvector of $T^k$ for some $k$, then $\mathbf{v}$ is an eigenvector of $T$.

Now, given a real polynomial, $p(t) = a_0 + a_1 t + a_2 t^2 + \cdots a_n t^n$, define the transformation $p(T) : V \to V$ by

$$P(T) = a_0 I + a_1 T + a_2 T^2 + \cdots a_n T^n$$

where $I : V \to V$ is the identity transformation.

(3) Show that, if $\mathbf{v} \in V$ is an eigenvector of $T$ with eigenvalue $\lambda$, then $\mathbf{v}$ is an eigenvector of $p(T)$ with eigenvalue

$$p(\lambda) = a_0 + a_1 \lambda + a_2 \lambda^2 + \cdots + a_n \lambda^n.$$

**Bonus:** In the case that $V$ is finite dimensional[b], one can define the linear transformation $e^T = \exp(T) : V \to V$ by the formula

$$\begin{aligned}\exp(T) &= I + T + \frac{1}{2!}T^2 + \frac{1}{3!}T^3 + \cdots \\ &= \sum_{k=0}^{\infty} \frac{1}{k!}T^k\end{aligned}$$

where $T^0 = I$ and $0! = 1$; this is done to emulate the Maclaurin series for $e^x$, i.e.,

$$\begin{aligned}e^x &= 1 + x + \frac{1}{2!}x^2 + \cdots \\ &= \sum_{k=0}^{\infty} \frac{x^k}{k!}\end{aligned}$$

defined for $x \in \mathbb{R}$. You do not need to worry about the convergence of the power series in $T$, you may just manipulate it formally and not worry about convergence tests or anything like that.

(4) Show that, if $\lambda$ is an eigenvalue of $T$, then $e^\lambda$ (the number) is an eigenvalue of $\exp(T)$.

(5) Using the above definition for $\exp(T)$ in terms of power series as an analogue, find a reasonable definition of $\cos(T) : V \to V$ and show that $\cos(\lambda)$ is an eigenvalue of this series whenever $\lambda$ is an eigenvalue of $T$.

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[a] By "show or find a counterexample", I mean do one of the following two things: 1) If the statement is true, show that it is true. 2) If the statement is false, present an example of a specific (with numbers!) linear transformation $T$ on a specific vector space $V$ and demonstrate that there is a vector $\mathbf{v}$ for which the conclusion of the statement does not hold.

[b] Or more generally, $V$ is a special type of vector space called a *Banach space* and $T : V \to V$ is a special type of transformation called *bounded*. These appear ubiquitously in differential equations and quantum mechanics. You do not need to know this vocabulary; I'm simply giving it for your cultural benefit.

## Appendix

This appendix amasses some basic facts about linear transformations that you might find useful to you – we discussed all of these in our discussion of linear transformations (both on $\mathbb{R}^n$ and abstract vector spaces).

**Definition 2.5.** *Let $V$ and $W$ be vector spaces and let $T : V \to W$ be a linear transformation. The transformation $T$ is said to be injective (or one-to-one) if the equation*

$$T(\mathbf{v}_1) = T(\mathbf{v}_2)$$

*only holds when $\mathbf{v}_1 = \mathbf{v}_2$. The transformation $T$ is said to be surjective (or onto $W$) if, for each $\mathbf{w} \in W$, there is some $\mathbf{v} \in V$ such that $T(\mathbf{v}) = \mathbf{w}$. Finally, if a linear map $T$ is injective and surjective, we say that $T$ is bijective (or that $T$ is an isomorphism, or a bijection, and in this case we also say $V$ and $W$ are isomorphic).*

In plain terms, the transformation $T$ is surjective if every element in $W$ is hit by $T$. To prove that a given linear transformation is surjective (if it is indeed surjective), you must take an arbitrary vector $\mathbf{w}$ in $W$ (whatever that looks like) and find some vector $\mathbf{v} \in V$ which is mapped to the vector $\mathbf{w}$ by $T$, i.e., $T(\mathbf{v}) = \mathbf{w}$. To show that $T$ is injective, the following characterization is often useful.

**Proposition 2.6.** *Let $T : V \to W$ be a linear transformation between vector spaces $V$ and $W$. Then $T$ is injective if and only if* $\ker(T) = \{\mathbf{v} \in V : T(\mathbf{v}) = \mathbf{0}_W\}$ *contains only the zero vector in $V$, i.e.,* $\ker(T)$ *is "trivial".*

*Proof.* Let's first assume that $T$ is injective and let $\mathbf{v} \in \ker(T)$. Our job is to show that $\mathbf{v} = \mathbf{0} = \mathbf{0}_V$. Because linear maps always map the zero vector $\mathbf{0}_V$ in $V$ to the zero vector $\mathbf{0}_W$ in $W$, we have

$$T(\mathbf{v}) = \mathbf{0}_W = T(\mathbf{0}_V)$$

and so, because $T$ is injective, $\mathbf{v} = \mathbf{0}_V$. Thus the zero vector (in $V$) is the only vector in the kernel of $T$.

Conversely, assume that $\ker(T) = \{\mathbf{0}_V\}$ and suppose that, for some $\mathbf{v}_1$ and (possibly distinct) $\mathbf{v}_2$ in $V$, we have

$$T(\mathbf{v}_1) = T(\mathbf{v}_2).$$

Then, by the linearity of $T$,

$$T(\mathbf{v}_1 - \mathbf{v}_2) = T(\mathbf{v}_1) - T(\mathbf{v}_2) = \mathbf{0}_W$$

and hence $\mathbf{v}_1 - \mathbf{v}_2 \in \ker(T)$. But, since the kernel of $T$ is trivial,

$$\mathbf{v}_1 - \mathbf{v}_2 = \mathbf{0}_V,$$

i.e., $\mathbf{v}_1 = \mathbf{v}_2$. Thus, we have shown that for any choice of $\mathbf{v}_1$ and $\mathbf{v}_2$ such that $T(\mathbf{v}_1) = T(\mathbf{v}_2)$, it must be true that $\mathbf{v}_1 = \mathbf{v}_2$. This is precisely what it means to say that $T$ is injective. □

Another useful proposition is stated as follows.

**Proposition 2.7.** *Let $T : V \to W$ be a linear transformation from $V$ to $W$ (real vector spaces). Then $T$ is bijective if and only if $T$ is invertible, i.e., there is a (unique) map $T^{-1} : W \to V$ for which $T \circ T^{-1} = I_W$ and $T^{-1} \circ T = I_V$; here, $I_V$ and $I_W$ denote the identity transformations on $V$ and $W$, respectively.*