

# Math 253 – Week of March 30th

## Problems To Chew On

- Problem 1.** (a) For each real number  $r$ , define the linear transformation  $D_r : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  by setting  $D_r(\mathbf{x}) = r\mathbf{x}$ .
- i. Find the standard matrix for  $D_r$ .
  - ii. Given two real numbers  $r$  and  $s$ , describe the composition  $D_r \circ D_s$ .
- (b) For each real number  $\theta$ , define the linear transformation  $R_\theta : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  by setting  $R_\theta(\mathbf{x}) = A_\theta\mathbf{x}$ , where  $A_\theta = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$ . Use matrix multiplication to verify that, for any real numbers  $\theta$  and  $\phi$ , we have  $R_\theta \circ R_\phi = R_{\theta+\phi}$  and explain why this makes sense geometrically.
- (c) For each real number  $a$ , define the linear transformation  $T_a : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  by setting  $T_a(\mathbf{x}) = M_a\mathbf{x}$ , where  $M_a = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$ .
- i. Describe what  $T_a$  does to the plane geometrically.
  - ii. Given two real numbers  $a$  and  $b$ , use matrix multiplication to find a simpler form for the composition  $T_a \circ T_b$ . Explain why your answer makes sense geometrically.

**Problem 2. A frolic through the land of reflections and rotations.**

Consider the linear transformations  $F_1 : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  and  $F_2 : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  with standard matrix representations

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

respectively.

- (a) In  $\mathbb{R}^2$ , plot the triangle with vertices  $\mathbf{v}_1 = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$ ,  $\mathbf{v}_2 = \begin{pmatrix} 4 \\ 0 \end{pmatrix}$  and  $\mathbf{v}_3 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$  and label each vertex.
- (b) To understand the linear transformations  $F_1$  and  $F_2$ , apply both maps to the vertices  $\mathbf{v}_1, \mathbf{v}_2$  and  $\mathbf{v}_3$ . Please plot your results in the same plane as your triangle from the previous part. You should obtain two triangles. Please label each of the six vertices as  $F_k(\mathbf{v}_j)$  for the appropriate values of  $k$  and  $j$ .
- (c) By studying your plot, you should see that transformations  $F_1$  and  $F_2$  are both *reflections*. In other words,  $F_1$  is a reflection over the line  $\ell_1$  and  $F_2$  is a reflection over the line  $\ell_2$ . What are  $\ell_1$  and  $\ell_2$ ?
- (d) You can compose the transformation  $F_1$  with  $F_2$  by first applying the map  $F_2$  to a vector  $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$  and then applying the the map  $F_1$  to the result. This produces a new linear transformation  $T = F_1 \circ F_2 : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ . Thinking about this transformation  $T$  diagrammatically, do you expect  $T$  to be another reflection? If so, over what line? If not, what is it? Please answer the question by drawing a new plot and following the triangle with vertices  $\mathbf{v}_1, \mathbf{v}_2$  and  $\mathbf{v}_3$  by first applying  $F_2$  and then applying  $F_1$  to the result.
- (e) Given a vector  $\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \in \mathbb{R}^2$  which has  $\|\mathbf{u}\| = \sqrt{u_1^2 + u_2^2} = 1$ , called a *unit vector*, let  $\mathbf{u}^\perp = \begin{pmatrix} u_2 \\ -u_1 \end{pmatrix}$ . Using  $\mathbf{u}$  and  $\mathbf{u}^\perp$ , we define a transformation  $\text{Ref}_{\mathbf{u}} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  by the formula

$$\begin{aligned} \text{Ref}_{\mathbf{u}}(\mathbf{x}) &= (\mathbf{u} \cdot \mathbf{x})\mathbf{u} - (\mathbf{u}^\perp \cdot \mathbf{x})\mathbf{u}^\perp \\ &= \left( \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right) \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} - \left( \begin{pmatrix} u_2 \\ -u_1 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right) \begin{pmatrix} u_2 \\ -u_1 \end{pmatrix} \end{aligned}$$

where  $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$  and  $\cdot$  denotes the dot product (from Homework 1). To understand this definition, compute the standard matrix representations for the transformations  $\text{Ref}_{\mathbf{e}}$  and  $\text{Ref}_{\mathbf{d}}$  where  $\mathbf{e} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$  and  $\mathbf{d} = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$ . Explain how these transformations relate to  $F_1$  and  $F_2$ .

- (f) It is not hard to see, for each unit vector  $\mathbf{u}$ ,  $\text{Ref}_{\mathbf{u}}$  represents a reflection about a line  $\ell$  in  $\mathbb{R}^2$ . Based on your response from the previous item, make a conjecture/guess which says how the line  $\ell$  relates to the vector  $\mathbf{u}$ .
- (g) Prove that, for an arbitrary unit vector  $\mathbf{u}$ ,  $\text{Ref}_{\mathbf{u}}$  is a linear transformation. Hint: You can use properties of the dot product (which should have appeared in your MA122/MA161-162 text). In working with the definition of  $\text{Ref}_{\mathbf{u}}$ , it is important to always keep track of which objects are vectors and which objects are scalars.
- (h) It is not hard to see that every unit vector can be written in the form  $\begin{pmatrix} \cos(\theta) \\ \sin(\theta) \end{pmatrix}$  for some angle  $\theta$ . What does  $\theta$  represent (geometrically) for the corresponding unit vector?
- (i) Writing  $\mathbf{u}_\theta = \begin{pmatrix} \cos(\theta) \\ \sin(\theta) \end{pmatrix}$ , compute the standard matrix representation for  $\text{Ref}_{\mathbf{u}_\theta}$ .
- (j) Based on your observations (much earlier) concerning  $T = F_1 \circ F_2$ , we expect that  $\text{Ref}_{\mathbf{u}_\theta} \circ \text{Ref}_{\mathbf{u}_\phi}$  to be a rotation in the plane. Thus, we should have

$$R_\gamma = \text{Ref}_{\mathbf{u}_\theta} \circ \text{Ref}_{\mathbf{u}_\phi}$$

for some angle  $\gamma$ . Using geometric reasoning (and providing a drawing), what is  $\gamma$  in terms of  $\theta$  and  $\phi$ ?

- (k) Justify your answer by computing  $\text{Ref}_{\mathbf{u}_\theta} \circ \text{Ref}_{\mathbf{u}_\phi}$  and simplifying.

**Problem 3.** Let  $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$  be a linear transformation and let  $A$  be its standard matrix representation, which is necessarily an  $m \times n$  matrix.

- (a) Using the theory we have developed so far in this class, prove the following statement:

*If  $T$  is both one-to-one and onto, then  $m = n$  and so the matrix  $A$  is square.*

- (b) Prove or give a counterexample of the following statement:

*If  $A$  is a square matrix (i.e.,  $m = n$ ), then  $T$  is one-to-one.*

- (c) Prove or give a counterexample of the following statement:

*If  $A$  is a square matrix (i.e.,  $m = n$ ), then  $T$  is onto.*

Note: By "find a counterexample", we mean to find an explicit square matrix (say a  $2 \times 2$  or  $3 \times 3$ ) for which the conclusion (one-to-one or onto) does not hold. Further, you should demonstrate your assertion. For example, if you can find a  $2 \times 2$  matrix whose corresponding linear transformation  $T$  is not one-to-one, give an example of a non-zero vector  $\mathbf{x}$  for which  $T(\mathbf{x}) = \mathbf{0}$ .

**Problem 4.** It is well known that for general matrices  $A$  and  $B$ ,  $AB \neq BA$ , even when both operations are defined. Here we will discuss what matrices  $A$  do have the property,  $AB = BA$  for every other matrix  $B$ .

- (a) Let  $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$  be the  $2 \times 2$  matrix with  $a, b, c, d \in \mathbb{R}$ . Compute  $AI_2$  and  $I_2A$  for  $I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ , the identity matrix. Does  $I_2$  commute with any matrix  $A$ ?
- (b) Now, we will explore what conditions are required for  $A$  to commute with any other  $B$ . Let  $B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ . Compute  $AB$  and  $BA$ .

- (c) What conditions on  $a, b, c, d$  are required for  $AB = BA$ ? Write  $A$  simplified with these conditions.
- (d) Let  $C = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ . Compute  $AC$  and  $CA$ . Again, what conditions will make these two matrices equal? Write this simplified  $A$ .
- (e) Let  $D = \begin{pmatrix} p & q \\ r & s \end{pmatrix}$ . Compute  $AD$  and  $DA$ .
- (f) Write your own theorem based on your observations here.
- (g) Make a conjecture on what kinds of  $n \times n$  matrices commute with every other  $n \times n$  matrix.