Math 253 - Homework 5

Due in class on Wednesday, March 11

Write your answers clearly and carefully, being sure to emphasize your answer and the key steps of your work. You may work with others in this class, but the solutions handed in must be your own. If you work with someone or get help from another source, give a brief citation on each problem for which that is the case.

Part I

While you are expected to complete all of these problems, do not hand in the problems in Part I. You are encouraged to write complete solutions and to discuss them with me or your peers. As extra motivation, some of these problems will appear on the weekly quizzes.

1. Practice Problems:
   (a) Section 1.8: 1, 2, 3
   (b) Section 1.9: 1 (it's not numbered but there's only one)
2. Exercises:
   (a) Section 1.8: 2, 4, 6, 7, 9, 12, 21, 24, 26, 28, 29, 30, 32
   (b) Section 1.9: 1, 2, 4, 6, 9, 11, 16, 17, 18, 23, 24, 33, 34, 35

Part II

Hand in each problem separately, individually stapled if necessary. Please keep all problems together with a paper clip.

1. Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. In each part below, either carefully prove the statement or provide a clear counterexample.
   (a) *If $\mathbf{v}_1, \ldots, \mathbf{v}_k \in \mathbb{R}^n$ are linearly independent, then $T(\mathbf{v}_1), \ldots, T(\mathbf{v}_k)$ are linearly independent as well.*
   (b) *If $\mathbf{v}_1, \ldots, \mathbf{v}_k \in \mathbb{R}^n$ are linearly dependent, then $T(\mathbf{v}_1), \ldots, T(\mathbf{v}_k)$ are linearly dependent as well.*
   (c) *If $\mathbf{v}_1, \ldots, \mathbf{v}_k \in \mathbb{R}^n$ span $\mathbb{R}^n$, then $T(\mathbf{v}_1), \ldots, T(\mathbf{v}_k)$ span $\mathbb{R}^m$ as well.*
   (d) *If $\mathbf{v}_1, \ldots, \mathbf{v}_k \in \mathbb{R}^n$ do not span $\mathbb{R}^n$, then $T(\mathbf{v}_1), \ldots, T(\mathbf{v}_k)$ don't span $\mathbb{R}^m$ either.*
2. Fix a number θ and define the function $T : \mathbb{R}^2 \to \mathbb{R}^2$ by sending $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ to

$$T(\mathbf{x}) = \begin{pmatrix} (\cos^2 \theta)x_1 + (\cos\theta \sin\theta)x_2 \\ (\cos\theta \sin\theta)x_1 + (\sin^2 \theta)x_2 \end{pmatrix}.$$

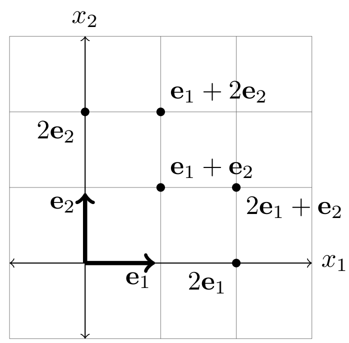
   (a) Show that T is a linear transformation.
   (b) Find the standard matrix for T.
   (c) In the case that $\theta = \pi/4$, describe the vectors $\mathbf{x} \in \mathbb{R}^2$ such that $T(\mathbf{x}) = \mathbf{0}$.
   (d) In the case of arbitrary θ, describe the vectors $\mathbf{x} \in \mathbb{R}^2$ such that $T(\mathbf{x}) = \mathbf{0}$.
   Note: Take care to address the separate cases when $\cos\theta$ or $\sin\theta$ is zero.
   (e) In the case that $\theta = \pi/4$, compute $T\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right)$, $T\left(\begin{pmatrix} 2 \\ 0 \end{pmatrix}\right)$, $T\left(\begin{pmatrix} 0 \\ 3 \end{pmatrix}\right)$, and $T\left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right)$.
   (f) What do you think T is doing to the plane? Explain your choice and illustrate with a picture.

3. (a) Describe in words the vector $\mathbf{v}_\theta = \begin{pmatrix} \cos\theta \\ 0 \\ \sin\theta \end{pmatrix}$ where $\theta \in \mathbb{R}$ (think rotations).

   (b) Let $\mathbf{u}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ and $\mathbf{u}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$. For which values of θ are the vectors $\mathbf{u}_1, \mathbf{u}_2, \mathbf{v}_\theta$ linearly independent? Describe how to interpret your answer geometrically.

   (c) Describe $\mathrm{Span}\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{v}_\theta\}$. Your answer should depend on θ.

4. Let $\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ in $\mathbb{R}^2$. Below is a diagram with seven vectors that are linear combinations of $\mathbf{e}_1$ and $\mathbf{e}_2$.



   (a) For the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$, where $T(\mathbf{x}) = A\mathbf{x}$ for the matrix $A = \begin{pmatrix} -1 & 0 \\ 1 & 2 \end{pmatrix}$, find $T(\mathbf{v})$ for each vector $\mathbf{v}$ in the diagram above.

   (b) Graph the image of the diagram above, labeling all the vectors you found in part (a).

   (c) Describe the transformation geometrically, using language from Section 1.9.

   (d) Consider another linear transformation $S : \mathbb{R}^2 \to \mathbb{R}^2$. If $S(\mathbf{e}_1 + \mathbf{e}_2) = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$, can you determine the image of $\mathbf{e}_1$ and $\mathbf{e}_2$? (I.e. what is $S(\mathbf{e}_1)$ and $S(\mathbf{e}_2)$?) If you can, do so. If not, give a counterexample. (I.e. produce two transformations with $S(\mathbf{e}_1 + \mathbf{e}_2) = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$, but different images of $\mathbf{e}_1$ and $\mathbf{e}_2$.)

   (e) Let $\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ and $\mathbf{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ in $\mathbb{R}^3$. Now let $P : \mathbb{R}^3 \to \mathbb{R}^2$ be the linear transformation defined by $P(\mathbf{x}) = A\mathbf{x}$ where $A = \begin{pmatrix} 4 & -3 & 1 \\ 3 & 1 & -2 \end{pmatrix}$. Find the image of $\mathbf{e}_1, \mathbf{e}_2$, and $\mathbf{e}_3$ and graph them in $\mathbb{R}^2$.

   (f) Describe why you could have guessed what the image of $\mathbf{e}_1, \mathbf{e}_2$, and $\mathbf{e}_3$ would be.

   (g) Show that P is onto.

   (h) Not every map $Q : \mathbb{R}^3 \to \mathbb{R}^2$ has to be onto. Find a linear transformation $Q : \mathbb{R}^3 \to \mathbb{R}^2$ such that Q is not onto and $Q(\mathbf{e}_1), Q(\mathbf{e}_2)$, and $Q(\mathbf{e}_3)$ are all distinct. Justify your answer and graph the images of $\mathbf{e}_1, \mathbf{e}_2$, and $\mathbf{e}_3$.