Math 253 - Homework 4

(not for handing in)

Write your answers clearly and carefully, being sure to emphasize your answer and the key steps of your work. You may work with others in this class, but the solutions handed in must be your own. If you work with someone or get help from another source, give a brief citation on each problem for which that is the case.

Part I

While you are expected to complete all of these problems, do not hand in the problems in Part I. You are encouraged to write complete solutions and to discuss them with me or your peers. As extra motivation, some of these problems will appear on the weekly quizzes.

1. Practice Problems:
   (a) Section 1.6: 1, 2
   (b) Section 1.7: 1-4
2. Exercises:
   (a) Section 1.6: 3, 4, 6, 8, 12, 14
   (b) Section 1.7: 1, 2, 7, 9, 15, 21, 27, 28, 31, 38

Suggested Review

As promised, below is a list of suggested review exercises followed by an itemized list of topics we've covered in Math 253 since the beginning of the semester. The list of topics (and the proportion of time we've spend on them since the beginning of the semester) will align with the problems you will see on Wednesday's midterm exam. In studying for Wednesday's midterm exam, please note that I consider the review exercises, homework exercises (both the "turn in" and "do not turn in") and the examples I've covered in lecture to be the best source of practice problems. If you know how to approach each problem, are able to work quickly and accurately, and understand the theory and methodology by which you have obtained a solution, you should perform well on the exam.

1. Review Exercises:
   (a) Section 1.1: 1, 11, 15, 25
   (b) Section 1.2: 3, 11, 18, 20, 30
   (c) Section 1.3: 8, 12, 15, 26, 34
   (d) Section 1.4: 6, 10, 11, 21, 30
   (e) Section 1.5: 5, 8, 15, 16, 21, 33

Study Topics (not an exhaustive list):

1. Systems of linear equations
   - You should know what linear equations and linear systems are. Specifically, you should know what a linear system of m equations in n variables is.

- You should know what is meant by a solution of a linear system and the multiple ways to write it, e.g., a collection of numbers $s_1, s_2, \ldots, s_n$ or in the form on an n-vector, $\begin{pmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{pmatrix}$.
- For a given linear system, you should know what a solution set is and what a general solution is. You should also understand the ways of expressing solution sets and general solutions, e.g., see the discussion surrounding Eq. (5) on Page 21.
- You should know what it means for two linear systems to be equivalent. You should think about why performing elementary row operations on a system (or augmented matrix) yields an equivalent system. Also, think about how this is the basis of problem solving via elementary row reduction.

2. Consistency/Inconsistency – Existence and uniqueness of solutions

   - You should know what it means for a linear system to be consistent or inconsistent. You should also have many many examples at the tips of your fingers concerning these notions.
   - When a system is consistent, you should know what it means for solutions to be unique. You should also know several ways of determining whether or not solutions are unique (more on this later).
   - You should know about (and have ample examples of) the possibilities concerning consistency/inconsistency/uniqueness/nonuniqueness. See, for example, the blue block on Page 4.

3. Augmented matrices, row operations, and echelon forms

   - You should know how to form the augmented matrix for a linear system.
   - You should know how to solve a linear system (or determine that it is inconsistent) performing elementary row operations (see Page 7).
   - You should know about row echelon form and reduced row echelon form.
   - Given a linear system, you should know how to form an augmented matrix for that system and perform a sequence of elementary row operations on that system to determine consistency/consistency/uniqueness. You should also be able to explain how sequences of row operations produced row equivalent matrices (and why these correspond to equivalent systems) and what this means for solution sets.
   - You should know Theorem 2 on Page 24.

4. Vector equations, linear combinations, span (as a noun *and* verb!)

   - You should know about $\mathbb{R}^n$, for a given integer n. You should know about vectors in $\mathbb{R}^n$ and the various operations on them (see the blue block on Page 32 – could you prove these properties?).
   - You should know what linear combinations are (and weights, etc.)
   - Given a list of vectors $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_p$, you should know that $\text{Span}\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_p\}$ is. This means understanding the (abstract) definition and being able to produce, on your own, many many examples of things in the span of some given vectors.
   - If $\mathbf{v} \in \text{Span}\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_p\}$, you should know what it means for $\mathbf{v}$ to be expressed *uniquely* as a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_p$.
   - You should know how to formulate linear systems of equations as vector equations. You should know (very very well, and be able to prove) the equivalence between vector equations and linear systems stated in the blue box on Page 34. This fact is equivalently stated as:

     Proposition 1. *Let $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ and $\mathbf{b}$ be vectors in $\mathbb{R}^m$. Then $\mathbf{b} \in \text{Span}\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\}$ if and only if the linear system with augmented matrix*

     $$[\mathbf{a}_1 \ \mathbf{a}_2 \ \ldots \ \mathbf{a}_n \ \mathbf{b}]$$

     has a solution.

- If you are given a list of vectors in $\mathbb{R}^n$, you should be able to determine if your list spans $\mathbb{R}^n$. If it does span $\mathbb{R}^n$, you should have a coherent argument/demonstration which shows that any vector in $\mathbb{R}^n$ can be written as a linear combination of your given vectors. If it doesn't span $\mathbb{R}^n$, you should be able to show that too.

5. Matrix equations

   - You should know how to formulate a linear system or vector equation as a matrix equation. You should understand the definition of the matrix-vector product.
   - Please know Theorem 3 on Page 42.
   - Please know Theorem 4 on Page 43 (and be able to prove it or portions of it).
   - You should know the basic (linear) properties of the matrix-vector product.

6. Homogeneous and non-homogeneous solutions

   - You should know how to define homogeneous and non-homogeneous linear systems. You should understand what these words mean in the context of matrix equations (and augmented matrices).
   - You should know that all homogeneous systems are consistent and what the trivial solution is.
   - You should understand (very very well and, perhaps, think about how you might prove) Theorem 6 on Page 53. In fact, to test if you understand this theorem, you should try to see if you can understand and perhaps prove the following corollary (a result whose proof follows easily from the theorem):

     Corollary 1. *Given an $m \times n$ matrix A and a vector $\mathbf{b} \in \mathbb{R}^m$, suppose that the matrix equation $A\mathbf{x} = \mathbf{b}$ is consistent. Then the solution to the matrix equation $A\mathbf{x} = \mathbf{b}$ is unique if and only if the homogeneous equation $A\mathbf{x} = \mathbf{0}$ admits only the trivial solution.*

7. Solution sets in vector parametric form

   - You should be able to express a solution set in parametric form.
   - You should know about how this parametric form comes about from solving linear systems and how the number of parameters coincides with the number of free variables.
   - You should have an idea of what the parametric representation of a solution set means, geometrically, for linear homogeneous or non-homogeneous systems.

8. Applications of linear systems

   - You've seen a number of applications on Homeworks 1, 2 and 3 of linear equations, systems, vector equations and matrix equations. You should understand these applications well.
   - You should understand the applications in Section 1.6 and the applications from the assigned book problems from that section.
   - Perhaps, most importantly, you should be very familiar with the temperature problem of Homework 1 and the chemical reaction application of Homework 3. For the exam, understanding the chemistry/physics of these examples, while fascinating, is less important the the "big ideas" in setting these problems up and realizing them as linear systems. Further, think about how the mathematical theory (all the stuff we're doing) yielded interesting implications for these application.

9. Linear dependence / independence

   - You should know (very very well) the definition of linear Independence/dependence for a list of vectors.
   - Given a list of vectors, you should be able to determine if they are linearly independent or dependent.

- You should understand (very well) that linear dependence (in the context of $\mathbb{R}^n$) is essentially a question about the uniqueness of solutions to a homogeneous system of the form $A\mathbf{x} = \mathbf{0}$. Can you set such a system up and solve it?
- Thinking back to Proposition 1, you should understand very very well (and be able to prove) the following refinement/follow-up proposition.

  Proposition 2. *Let* $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ *and* $\mathbf{b}$ *be vectors in* $\mathbb{R}^m$. *Then* $\mathbf{b} \in \operatorname{Span}\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\}$ *if and only if the linear system with augmented matrix*

  $$[\mathbf{a}_1 \ \mathbf{a}_2 \ \ldots \ \mathbf{a}_n \ \mathbf{b}]$$

  has a solution.

  In the case that one (and hence both) of the above conditions are specified, the following are equivalent:

  (a) The vectors $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ *are linearly independent.*

  (b) $\mathbf{b}$ *is uniquely expressible as a linear combination of* $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$, *i.e., the weights used to write* $\mathbf{b}$ *as a linear combination of* $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ *are unique.*

  (c) The solution to the linear system with augmented matrix

  $$[\mathbf{a}_1 \ \mathbf{a}_2 \ \ldots \ \mathbf{a}_n \ \mathbf{b}]$$

  is unique.

- You should understand (but not over-extrapolate) the results concerning the linear independence/dependence of a list of one or two non-zero vectors.
- You should understand Theorem 7 on Page 68 and Theorems 8 and 9 on Page 69.