Math 253 - Homework 3

Due in class on Wednesday, February 26

This is the second homework assignment for Math 253 and it is broken into two parts. The first part of the homework consists of exercises you should do (and I'll expect you to do) but you needn't turn in. As these exercises will not be graded, if you would like help with them or just want to make sure you're doing them correctly, you should (always) feel free to come to office hours (mine or those of the TAs). As extra motivation, some of the problems from Part I will appear on the weekly quizzes. The second part is the part you are expected to turn in. More precisely, please complete all problems in Part II, write up clear and thorough solutions for them (consistent with the directions given in the syllabus) and hand them in. You may work with others in this class, but the solutions handed in must be your own. If you work with someone or get help from another source, give a brief citation on each problem for which that is the case. Your write-ups are due on Wednesday, February 19th at the beginning of class. As always, please come and see me early if you get stuck on any part of this assignment. I am here to help!

Part I: Do Not Turn In

While you are expected to complete all of these problems, do not hand in the problems in Part I.
You are encouraged to write complete solutions and to discuss them with me or your peers.
As extra motivation, some of these problems will appear on the weekly quizzes.

1. Practice Problems:
   (a) Section 1.4: 1, 2
   (b) Section 1.5: 1, 2
2. Exercises:
   (a) Section 1.4: 1-5, 9, 13, 15, 23, 25
   (b) Section 1.5: 1, 7, 11, 13, 17

Part II

Hand in each problem separately, individually stapled if necessary.
Please keep all problems together with a paper clip.

1. Let A be the following 3×5 matrix:

$$A = \begin{pmatrix} 1 & -4 & 0 & 2 & 1 \\ -3 & 12 & 2 & -3 & 0 \\ -1 & 4 & 1 & -1 & 0 \end{pmatrix}$$

In each of the following parts, be sure to give any solutions of systems in parametric vector form.

(a) Solve the homogeneous system $A\mathbf{x} = \mathbf{0}$.

(b) Let $\mathbf{b}_1 = \begin{pmatrix} 7 \\ 2 \\ 0 \end{pmatrix}$. Find *one* solution to the system $A\mathbf{x} = \mathbf{b}_1$.

(c) Solve the non-homogeneous system $A\mathbf{x} = \mathbf{b}_1$.
Hint: your answer should follow immediately from parts (a) and (b).

(d) Let $\mathbf{b}_2 = \begin{pmatrix} 100 \\ 100 \\ 100 \end{pmatrix}$. Find *one* solution to the system $A\mathbf{x} = \mathbf{b}_2$.

(e) Solve the non-homogeneous system $A\mathbf{x} = \mathbf{b}_2$.

(f) Let $\mathbf{b}_3 = \begin{pmatrix} 107 \\ 102 \\ 100 \end{pmatrix}$. Solve the non-homogeneous system $A\mathbf{x} = \mathbf{b}_3$.

2. Consider the linear equation $y = 3x - 2$. In the x, y-plane, $\mathbb{R}^2$, this equation represents a line.
   (a) Describe the solution set to $y = 3x - 2$ in $\mathbb{R}^2$ as a parametric vector equation, $\mathbf{x} = t\mathbf{v}_1 + \mathbf{p}_1$, for $t \in \mathbb{R}$.
   (b) Provide another parametric vector equation, $\mathbf{x} = s\mathbf{v}_2 + \mathbf{p}_2$, for $s \in \mathbb{R}$, which is also a solution to $y = 3x - 2$ in $\mathbb{R}^2$, and where $\mathbf{p}_1 \neq \mathbf{p}_2$ and $\mathbf{v}_1 \neq \mathbf{v}_2$.
   (c) Show that the parametric vector equation you found in part (b) also satisfies the equation $y = 3x-2$.
   (d) Now consider the same equation $y = 3x - 2$, but in $\mathbb{R}^3$. Describe the solution set as a parametric vector equation, $\mathbf{x} = t\mathbf{v} + s\mathbf{u} + \mathbf{p}_3$, for $t, s \in \mathbb{R}$.
   (e) What can be said about the solution sets in part (a) compared to the geometry in part (d)?

3. Consider the following chemical equations. See Section 1.6 for a discussion on this topic.
   (a) Below is the chemical equation for the combustion of pentane, C_5H_{12}. Write down a vector equation for the chemical equation.
   $$C_5H_{12} + O_2 \to CO_2 + H_2O$$
   (b) Solve the vector equation in part (a). Write your answer both as a general solution and in parametric vector form.
   (c) Provide a minimal solution of the chemical equation properly balanced, with positive integer coefficients. Why do we require the coefficients to be positive?
   (d) Write down a matrix equation, but do *not* solve, for the following chemical equation.
   $$Al_2(CO_3)_3 + H_3PO_4 \to AlPO_4 + CO_2 + H_2O$$
   (e) Do we expect the columns of a matrix obtained from a chemical equation to be linearly independent? Explain your reasoning.

4. Define two vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^3$ as follows:
   $$\mathbf{u} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, \quad \mathbf{v} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$
   (a) Do $\mathbf{u}$ and $\mathbf{v}$ span $\mathbb{R}^3$? Justify your answer with an explicit computation and explain why you could have anticipated this outcome.
   (b) Find a nonzero vector $\mathbf{w}_1$ which lies in the span of $\mathbf{u}$ and $\mathbf{v}$. If A is the 3×2 matrix with $\mathbf{u}$ and $\mathbf{v}$ as its columns, then what can you say about the matrix equation $A\mathbf{x} = \mathbf{w}_1$?
   (c) Do $\mathbf{u}, \mathbf{v}$, and $\mathbf{w}_1$ span $\mathbb{R}^3$? Again, support your answer with an explicit computation.
   (d) Prove the following general statement:
   $$\textit{If } \mathbf{b} \textit{ is in the span of } \mathbf{v}_1, \ldots, \mathbf{v}_k, \textit{ then } \mathit{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_k\} = \mathit{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_k, \mathbf{b}\}.$$
   Hint: To prove that two spans are the same, show that if $\mathbf{x}$ *belongs to the first, then it must belong to the second, and vice versa.*
   (e) Find a nonzero vector $\mathbf{w}_2$ which *does not* lie in the span of $\mathbf{u}$ and $\mathbf{v}$. Justify your choice by computing the solutions to $A\mathbf{x} = \mathbf{w}_2$, where A is defined as in part (b).
   (f) Show that $\mathbf{u}, \mathbf{v}$, and $\mathbf{w}_2$ span $\mathbb{R}^3$.

5. Given an angle θ, consider the matrix

$$R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

(a) For a general vector $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, compute $R(\theta)\mathbf{x}$. You should observe that this defines a function from $\mathbb{R}^2$ to $\mathbb{R}^2$. Have you seen this function before? Explain and make sure you take into account the distinction between the matrix and the function.

(b) Given what you know about the function $\mathbf{x} \mapsto R(\theta)\mathbf{x}$, for an arbitrary vector $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$, do you expect solutions to the equation

$$R(\theta)\mathbf{x} = \mathbf{b}$$

to exist and be unique? Confirm your answer by first solving/reducing the homogeneous system $R(\theta)\mathbf{x} = \mathbf{0}$ and then considering the non-homogeneous case.

(c) Let α and β be two arbitrary (but fixed) angles and consider again the vector $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$. Without doing any computations (using only your intuition about $R(\theta)$), describe the vector[1] $R(\alpha)R(\beta)\mathbf{x} = R(\alpha)(R(\beta)\mathbf{x})$. Do you expect that $R(\alpha)R(\beta)\mathbf{x} = R(\gamma)\mathbf{x}$ for some angle γ? If so, what should γ be in terms of α and β?

(d) By doing the explicit computation, making use of trigonometric identities, confirm your answer from Part (c).

6. Given an angle θ, consider the 3×3 matrix

$$R_z(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

(a) Set $\mathbf{e} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ and $\mathbf{d} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ and compute $R_z(\theta)\mathbf{e}$ and $R_z(\theta)\mathbf{d}$.

(b) For any θ of your choosing, draw the vectors $\mathbf{e}$, $\mathbf{d}$, $R_z(\theta)\mathbf{e}$, and $R_z(\theta)\mathbf{d}$ in $\mathbb{R}^3$. I know drawing in 3D isn't easy, but please try your best. What happened geometrically to the vectors $\mathbf{e}$ and $\mathbf{d}$ when the matrix $R_z(\theta)$ was applied?

(c) Assume once more that θ is arbitrary but fixed. In view of your observations in Part (b), without doing any computations, describe what the matrix $R_z(\theta)$ does (or should do) to any vector of the form $\begin{pmatrix} a \\ b \\ 0 \end{pmatrix}$. Does the application of the matrix $R_z(\theta)$ "lift" the vector $\begin{pmatrix} a \\ b \\ 0 \end{pmatrix}$ out of the x_1x_2-plane?

(d) Compute $R_z(\theta)\begin{pmatrix} a \\ b \\ 0 \end{pmatrix}$ and confirm that your answer in the previous part makes sense. If useful, you may reference your conclusions from the previous problem.

(e) Compute $R_z(\theta)\begin{pmatrix} a \\ b \\ c \end{pmatrix}$. In view of this computation and your answers in the previous part, make a general statement as to what the matrix $R_z(\theta)$ does to a general vector $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ in $\mathbb{R}^3$.

[1] By this we mean the vector produced by applying the matrix $R(\alpha)$ to the vector $R(\beta)\mathbf{x}$.

(f) Let's now think about vectors that are "preserved" by the matrix $R_z(\theta)$. To this end, we seek a vector $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ for which

$$R_z(\theta)\mathbf{x} = \mathbf{x}. \tag{1}$$

To find such a vector, it is useful to write out the matrix product $R_z(\theta)\mathbf{x}$ and subtract from it the vector $\mathbf{x}$. This should transform Equation (1) into a homogeneous system of the form

$$A\mathbf{x} = \mathbf{0} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \tag{2}$$

What is the matrix A?

(g) Solve the homogeneous system (2). Does your answer depend on θ?

(h) Any solution $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ gotten in the previous part, by design, should satisfy Equation (1). Verify that this is true and, based on your geometric understanding of the matrix $R_z(\theta)$, explain why your answer makes sense.