Math 253 - Homework 2

Due in class on Wednesday, February 19

This is the second homework assignment for Math 253 and it is broken into two parts. The first part of the homework consists of exercises you should do (and I'll expect you to do) but you needn't turn in. As these exercises will not be graded, if you would like help with them or just want to make sure you're doing them correctly, you should (always) feel free to come to office hours (mine or those of the TAs). As extra motivation, some of the problems from Part I will appear on the weekly quizzes. The second part is the part you are expected to turn in. More precisely, please complete all problems in Part II, write up clear and thorough solutions for them (consistent with the directions given in the syllabus) and hand them in. You may work with others in this class, but the solutions handed in must be your own. If you work with someone or get help from another source, give a brief citation on each problem for which that is the case. Your write-ups are due on Wednesday, February 19th at the beginning of class. As always, please come and see me early if you get stuck on any part of this assignment. I am here to help!

Part I: Do Not Turn In

While you are expected to complete all of these problems, do not hand in the problems in Part I. You are encouraged to write complete solutions and to discuss them with me or your peers. As extra motivation, some of these problems will appear on the weekly quizzes.

1. Practice Problems:
   (a) Section 1.3: 1, 2
2. Exercises:
   (a) Section 1.3: 1-13 odd, 19, 21, 25, 27, 29, 33

Part II

Hand in each problem separately, individually stapled if necessary. Please keep all problems together with a paper clip.

1. The *center of mass* of $\mathbf{v}_1, \ldots, \mathbf{v}_k$ in $\mathbb{R}^n$, with a mass of m_i at $\mathbf{v}_i$, for $i = 1, \ldots, k$ and total mass $m = m_1 + \cdots + m_k$, is given by

$$\bar{\mathbf{v}} = \frac{m_1\mathbf{v}_1 + \cdots + m_k\mathbf{v}_k}{m} = \frac{m_1}{m}\mathbf{v}_1 + \cdots + \frac{m_k}{m}\mathbf{v}_k.$$

The *centroid*, which can be thought of as the geometric center, of $\mathbf{v}_1, \ldots, \mathbf{v}_k$ in $\mathbb{R}^n$ is the center of mass with $m_i = 1$ for each $i = 1, \ldots, k$, i.e.

$$\bar{\mathbf{c}} = \frac{\mathbf{v}_1 + \cdots + \mathbf{v}_k}{k} = \frac{1}{k}\mathbf{v}_1 + \cdots + \frac{1}{k}\mathbf{v}_k.$$

   (a) Let $\mathbf{v}_1 = \begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} -1 \\ 0 \\ 5 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 1 \\ 3 \\ -3 \end{bmatrix}, \mathbf{v}_4 = \begin{bmatrix} -5 \\ 0 \\ 2 \end{bmatrix}, \mathbf{v}_5 = \begin{bmatrix} 1 \\ -4 \\ -5 \end{bmatrix}$. Let $\mathbf{u}_k = \sum_{i=1}^{k} \mathbf{v}_i$. Compute each $\mathbf{u}_i$ for $i = 1, \ldots, 5$.

   (b) Compute the centroid of the set $\{\mathbf{v}_1, \ldots, \mathbf{v}_5\}$ and again of the set $\{\mathbf{u}_1, \ldots, \mathbf{u}_5\}$. Explain why the the centroid of the $\mathbf{v}_i$'s might have been expected.

   (c) Interpret the centroid of the $\mathbf{u}_i$'s as a linear combination of the $\mathbf{v}_i$'s. Is this realizable as the center of mass of the $\mathbf{v}_i$'s for some masses m_i?

(d) Set up and solve a system of equations to find a weight assignment for $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ such that the center of mass of these three vectors is at $\bar{\mathbf{v}} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. How many solutions to the system are there?

Does this accurately describe the number of possible masses which will produce $\bar{\mathbf{v}} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$? Explain your reasoning.

2. (a) Let $\mathbf{u} = \begin{bmatrix} 1 \\ -5 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$. Show that for any a and b, the vector $\begin{bmatrix} a \\ b \end{bmatrix}$ is in Span$\{\mathbf{u}, \mathbf{v}\}$.

(b) Let $\mathbf{u} = \begin{bmatrix} 1 \\ -5 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} -2 \\ a \end{bmatrix}$. For which values of a is it true that the vector $\begin{bmatrix} 777 \\ 9311 \end{bmatrix}$ is in Span$\{\mathbf{u}, \mathbf{v}\}$? Interpret your answer geometrically.

(c) Let $\mathbf{u} = \begin{bmatrix} 1 \\ -5 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} -2 \\ a \end{bmatrix}$. For which values of a and b is it true that the vector $\begin{bmatrix} 3 \\ b \end{bmatrix}$ is in Span$\{\mathbf{u}, \mathbf{v}\}$?

(d) Let $\mathbf{u} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$. Give an explicit description of Span$\{\mathbf{u}, \mathbf{v}\}$ in terms of θ and interpret your answer geometrically.

3. (a) A system of linear equations is *underdetermined* if it has fewer equations than variables. Discuss the possible sizes of the solution set for an underdetermined system; for each possibility, give an augmented matrix in reduced echelon form which illustrates your claim.

(b) A system of linear equations is *overdetermined* if it has more equations than variables. Discuss the possible sizes of the solution set for an overdetermined system; for each possibility, give an augmented matrix in reduced echelon form which illustrates your claim.

(c) A system of linear equations is *homogeneous* if each equation has a constant term of zero (i.e. each entry in the augmentation column of the corresponding matrix is zero). Discuss the possible sizes of the solution set for a homogeneous system; for each possibility, give an augmented matrix in reduced echelon form which illustrates your claim.

4. A common way to describe a line in 3-dimensions is with parametric equations for x, y and z. Consider the system of parametric equations which define a line $\ell(t)$:

$$\ell(t) = \begin{cases} x(t) = & 3t - 4 \\ y(t) = & -t - 1 \\ z(t) = & 2t + 1 \end{cases}$$

where t is any real number.

(a) Find a system of two equations in three unknowns which represents two distinct planes whose intersection is ℓ.

(b) Find a system of three equations in three unknowns which represents three distinct planes whose intersection is ℓ. These planes should be all be distinct from your answers from Part (a).

(c) Select one equation from your answer in Part (a) and one equation from your answer in Part (b). Make an educated guess as to what the intersection of their corresponding planes should be. Explain.

(d) Verify your guess in part (b) by finding the intersection of the pair of planes you chose.

5. Consider the vectors

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \qquad \mathbf{v}_2 = \begin{pmatrix} 1 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \qquad \mathbf{v}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \qquad \mathbf{v}_4 = \begin{pmatrix} 0 \\ 2 \\ 0 \\ 1 \end{pmatrix}, \qquad \text{and} \qquad \mathbf{b} = \begin{pmatrix} 4 \\ 0 \\ 0 \\ -2 \end{pmatrix}.$$

(a) Show that $\mathbf{b}$ is a member of $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$. Are the weights for your linear combination unique? In other words, is there only one choice of scalars c_1, c_2, c_3, and c_4 for which $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 + c_4\mathbf{v}_4 = \mathbf{b}$?

(b) Show that $\mathbf{b}$ is a member of $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$. Are the weights for your linear combination unique?

(c) To understand the distinction between the two preceding items, we introduce an important concept which will follow us throughout the course:

Definition 1. *Let $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_p$ be vectors in $\mathbb{R}^n$ and let $\mathbf{0}$ denote the zero vector. We say that the vectors $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_p$ are linearly independent if the equation*

$$c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_p\mathbf{u}_p = \mathbf{0}$$

can only be satisfied if $c_1, c_2, \ldots, c_p$ are all zero.

Show that the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$ are **not** linearly independent[1].

(d) Show that the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are linearly independent.

(e) Returning to the general picture, let $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_p$ be vectors in $\mathbb{R}^n$ and let $\mathbf{b} \in \text{Span}\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_p\}$. Prove (i.e. give a convincing argument for) the following proposition.

Proposition 1. *If $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_p$ are linearly independent, then there is a unique (only one) choice of weights $c_1, c_2, \ldots, c_p$, for which*

$$\mathbf{b} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_p\mathbf{u}_p.$$

Hint: You may want to suppose that $\mathbf{b}$ can be expressed using two (possibly) different sets of weights $c_1, c_2, \ldots, c_p$ and $a_1, a_2, \ldots, a_p$. Then use linear independence to show that the weights all must be the same.

(f) Finally, explain why your conclusion for Part (a) was different from your conclusion for Part (b). Your explanation should be no more than two sentences and it should reference the proposition above.

6. Given vectors $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ and $\mathbf{b}$ in $\mathbb{R}^m$, consider the matrix equation

$$A\mathbf{x} = [\mathbf{a}_1 \ \mathbf{a}_2 \ \cdots \ \mathbf{a}_n] \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \mathbf{b}. \tag{1}$$

We note, in view of the definition on Page 41 of the text, the matrix product above is defined to be

$$A\mathbf{x} = [\mathbf{a}_1 \ \mathbf{a}_2 \ \cdots \ \mathbf{a}_n] \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n.$$

[1]Unsurprisingly, such a collection is said to be linearly dependent.

(a) Show that Equation (1) has a solution $\mathbf{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ if and only if $\mathbf{b} \in \operatorname{Span}\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\}$.

(b) In view of the previous question (specifically the proposition therein), formulate a precise (and true) statement concerning the uniqueness of solutions to Equation (1) and linear independence/dependence of the vectors $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$.