Math 253 - Homework 1

Due in class on Wednesday, February 12

This is the first homework assignment for Math 253 and it is broken into two parts. The first part of the homework consists of exercises you should do (and I'll expect you to do) but you needn't turn in. As these exercises will not be graded, if you would like help with them or just want to make sure you're doing them correctly, you should (always) feel free to come to office hours (mine or those of the TAs). As extra motivation, some of the problems from Part I will appear on the weekly quizzes. The second part is the part you are expected to turn in. More precisely, please complete all problems in Part II, write up clear and thorough solutions for them (consistent with the directions given in the syllabus) and hand them in. You may work with others in this class, but the solutions handed in must be your own. If you work with someone or get help from another source, give a brief citation on each problem for which that is the case. Your write-ups are due on Wednesday, February 12th at the beginning of class. As always, please come and see me early if you get stuck on any part of this assignment. I am here to help!

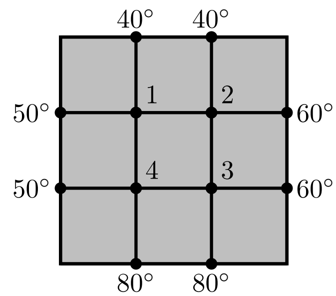
Part I: Do Not Turn In

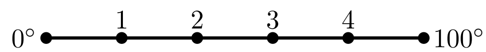
1. Practice Problems:
   (a) Section 1.1: 1, 2, 3, 4
   (b) Section 1.2: 1, 2
2. Exercises:
   (a) Section 1.1: 4, 7, 8, 16, 18, 26
   (b) Section 1.2: 2, 4, 8

Part II: Turn In

Hand in each problem separately, individually stapled if necessary.
Please keep all problems together with a paper clip.

1. In each part below, give an example of a system of two linear equations in two variables with the given property and show why your answer is correct:
   (a) The system has no solution.
   (b) The system has infinitely many solutions.
   (c) The system has a unique solution.
2. A thin metal plate is heated from the sides at varying temperatures. Assume heat flow is negligible in the direction perpendicular to the plate. Let T_1, T_2, T_3, T_4 denote the temperatures of the four nodes in the interior of the plate, corresponding to location 1,2,3,4, respectively. Assume the temperature of the node is the average of the four nearest nodes – to the left, above, right, and below.
   (a) Write a system of four equations whose solution gives estimates for the temperatures T_1, T_2, T_3, T_4 .
   (b) Solve this system of equations.
   (c) Repeat parts (a) and (b) for a metal wire, with four nodes evenly distributed in the interior and one end heated. Does your solution match your intuition?





3. A simple model of voter trends can be represented by a function from $\mathbb{R}^3$ to $\mathbb{R}^3$. For example, consider the function $M : \mathbb{R}^3 \to \mathbb{R}^3$ given by

$$M\begin{pmatrix} R \\ D \\ I \end{pmatrix} = \begin{pmatrix} \frac{7}{10}R + \frac{1}{10}D + \frac{3}{10}I \\ \frac{2}{10}R + \frac{8}{10}D + \frac{3}{10}I \\ \frac{1}{10}R + \frac{1}{10}D + \frac{2}{5}I \end{pmatrix}.$$

Here, R, D and I represent the number of Republicans, Democrats and Independents in a community, respectively. We call the vector $\begin{pmatrix} R \\ D \\ I \end{pmatrix} \in \mathbb{R}^3$ a *distribution* of voters. Given a distribution of voters $\begin{pmatrix} R \\ D \\ I \end{pmatrix}$, an application of the map M represents a new distribution of voters in a community after some period of time, say between presidential elections. In other words, M models the so-called drift of voters between party lines in a four-year period.

   (a) Suppose that, in 2020, there are 160 Republicans, 100 Democrats and 20 Independents, compute the number of Democrats, Republicans and Independents in 2024.

   (b) How many Republicans, Democrats and Independents will there be in 2028? Give a one sentence explanation about how your numbers were obtained from the original numbers $\begin{pmatrix} 160 \\ 100 \\ 20 \end{pmatrix}$ from 2020 and feel free to use your calculus vocabulary.

   (c) Suppose that, instead, in 2024, there are 90 Republicans, 150 Democrats and 40 independents. Find the number (previous) distribution of voters in 2020. This requires that you solve a 3×3 system of equations.

   (d) With the distribution you found in the previous part, without doing any computations, what will the distribution be in 2028? What will it be in 2044? Clearly explain your reasoning.

4. (a) The equations $3x - 6y + 3z = 9$ and $2x + y - 8z = 11$ describe a pair of planes in three dimensions. Using a system of linear equations, show that these planes intersect and carefully describe their intersection.

   (b) The equations $3x - 6y + 3z = 9$ and $2x - 4y + 2z = 5$ describe a pair of planes in three dimensions. Using a system of linear equations, show that these planes do not intersect. Explain how these planes are related to one another geometrically.

   (c) Consider the lines $2x - ay = 1$ and $6x + 3y = 4$, where a is a fixed constant. For which values of a do the two lines intersect? Use a system of linear equations to find your answer and then explain what it means geometrically.

5. Consider the augmented matrix $\begin{pmatrix} -3 & 1 & -1 \\ 2 & -4 & -1 \end{pmatrix}$.

   (a) What is the solution set to the corresponding system of linear equations?

   (b) Graph the lines corresponding to the system in the matrix.

   (c) Replace the first row with the sum of both rows. How does the graph change, and how does it stay the same? Verify the solution set does not change.

6. Recall from Math 122 (or its equivalent) that two non-zero vectors $\begin{pmatrix} a \\ b \end{pmatrix}$ and $\begin{pmatrix} c \\ d \end{pmatrix}$ were orthogonal (or perpendicular) if their dot product was zero, i.e., if $\begin{pmatrix} a \\ b \end{pmatrix} \cdot \begin{pmatrix} c \\ d \end{pmatrix} = ac + bd = 0$. In this case we wrote $\begin{pmatrix} a \\ b \end{pmatrix} \perp \begin{pmatrix} c \\ d \end{pmatrix}$.

   (a) Draw the vectors $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 2 \end{pmatrix}$ in $\mathbb{R}^2$ and, using the dot product, check if they are orthogonal. Given your drawing, does your conclusion make sense?

   (b) Consider the function $T : \mathbb{R}^2 \to \mathbb{R}^2$ given by

   $$T \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \frac{a}{\sqrt{2}} + \frac{b}{\sqrt{2}} \\ -\frac{a}{\sqrt{2}} + \frac{b}{\sqrt{2}} \end{pmatrix}.$$

   As it turns out, this is a very special function that sends orthogonal vectors to orthogonal vectors and we will study these later in the course. Verify that $T \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is orthogonal to $T \begin{pmatrix} 2 \\ 2 \end{pmatrix}$.

   (c) Verify the more general statement: If $\begin{pmatrix} a \\ b \end{pmatrix} \perp \begin{pmatrix} c \\ d \end{pmatrix}$, then $T \begin{pmatrix} a \\ b \end{pmatrix} \perp T \begin{pmatrix} c \\ d \end{pmatrix}$.

   (d) A counter-clockwise rotation in $\mathbb{R}^2$ by angle θ is defined as a function $R_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ given by

   $$R_\theta \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \cos(\theta)a - \sin(\theta)b \\ \sin(\theta)a + \cos(\theta)b \end{pmatrix}.$$

   You should verify for yourself that $R_{-\pi/4} = T$ and so T is a counter-clockwise rotation by $\theta = -\pi/4$ (equivalently, a clockwise rotation by $\pi/4$). By choosing an angle θ (you choose your own specific θ which should be different from $-\pi/4$), show geometrically that the vector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is rotated by the angle θ by the function R_θ, i.e., show that $R_\theta \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is a vector which makes an angle θ with the vector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

   (e) Given your work on the previous item, you can take for granted that R_θ actually takes a general vector $\begin{pmatrix} a \\ b \end{pmatrix}$ and rotates it in the plane by an angle θ landing at the new vector $R_\theta \begin{pmatrix} a \\ b \end{pmatrix}$. Using your knowledge of high school geometry, argue the statement: If $\begin{pmatrix} a \\ b \end{pmatrix} \perp \begin{pmatrix} c \\ d \end{pmatrix}$, then $R_\theta \begin{pmatrix} a \\ b \end{pmatrix} \perp R_\theta \begin{pmatrix} c \\ d \end{pmatrix}$.

   (f) Recalling that the size or magnitude of a vector $\begin{pmatrix} a \\ b \end{pmatrix}$ in $\mathbb{R}^2$ is defined to be $\sqrt{a^2 + b^2}$, investigate (and make the most precise and general statement you can) concerning what happens to the size of the vector when you apply a rotation to it.