As promised, I have written an itemized list of topics we've covered in Math 165 since Midterm 2. As I mentioned in class, the final exam will be cumulative but will emphasize (approximately 75%) the material since Midterm 2. As you study for the final exam, you should put this list together with the previous two lists (for midterms 1 and 2) as you study for the final to form a comprehensive list. As always, in studying for this final, note that I consider the homework exercises and everything I've covered in lecture to be the best source of practice (problems, proof, etc.). If you know how to approach each problem/exercise/proof, are able to work quickly and accurately, and understand the theory and methodology by which you have obtained a solution/proof, you should perform well on the exam.

Definitions:

The following list enumerates all the definitions you need to know by heart. In particular, you should make sure to know all quantifiers involved in the definitions and the order in which they appear. Also, for each definition, you should be able to come up with several examples satisfying the definition (and hopefully things that don't satisfy the definition).

1. As it appears on the list of topics for the previous midterm exam, you should know the definition of differentiability of a function $F : \mathcal{D} \subseteq \mathbb{R}^n \to \mathbb{R}^m$.

2. For special cases where F is scalar-valued, i.e., $f : \mathcal{D} \subseteq \mathbb{R}$, the definition of differentiability can be seen using the gradient ∇f. You should know what the gradient is, how it appears in the definition of differentiability, i.e.,
$$f(\mathbf{x}+\mathbf{h}) = f(\mathbf{x}) + \nabla f(\mathbf{x}) \cdot \mathbf{h} + \|\mathbf{h}\|\mathcal{E}(\mathbf{h})$$
where $\mathcal{E}$ is a continuous function with $\mathcal{E}(\mathbf{0}) = 0$.

3. You should know the interperetation of the gradient as a vector that points in the direction of largest increase (and, as we'll discuss later, results that justify this).

4. You should know the definition of higher-order partial derivatives. In particular, it would be helpful to understand know the notations
$$\frac{\partial^2}{\partial x \partial y} = f_{yx}$$
for a function f of two variables, x, y. For a multi-index $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_3) \in \mathbb{N}^d$, we defined
$$D^\alpha f = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \frac{\partial^{\alpha_2}}{x_2^{\alpha_2}} \cdots \frac{\partial^{\alpha_d}}{x_d^{\alpha_d}} f = \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \cdots \partial_{x_d}^{\alpha_d}}$$
where $|\alpha| = \alpha_1 + \alpha_2 + \cdots \alpha_d$. Note: You should not confuse the notation $D_{\mathbf{v}}$ for directional derivatives with the above notation D^α for so-called differential operators.

5. For a real-valued function f which is n-times continuously differentiable at a point $\mathbf{x}_0$, you should know that its nth-order Taylor Polynomial at $\mathbf{x}_0$ is given by
$$T_{n,f,\mathbf{x}_0}(\mathbf{x}) = \sum_{|\alpha| \le n} \frac{D^\alpha f(\mathbf{x}_0)}{\alpha!} (\mathbf{x} - \mathbf{x}_0)^\alpha$$
where we sum over all multindices $\alpha \in \mathbb{N}^d$ for which
$$|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_d \le n;$$
for any such $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_d)$, $\alpha! = \alpha_1!\alpha_2! \cdots \alpha_d!$ and, for $\mathbf{x} - \mathbf{x}_0 = \mathbf{h} = (h_1, h_2, \dots, h_d)$,
$$(\mathbf{x} - \mathbf{x}_0)^\alpha = \mathbf{h}^\alpha = h_1^{\alpha_1} h_2^{\alpha_2} \cdots h_d^{\alpha_d}$$
To understand this, you should try to write it out for a relatively simple function f of two variables.

6. For a function f of two variables, you should know that the above expansion in the case that $n = 2$ can be expressed equivalently by

$$T_{2,f,\mathbf{x}_0}(x, y) = f(\mathbf{x}_0) + \nabla f(\mathbf{x}) \cdot \mathbf{h} + \frac{1}{2}\mathbf{h} \cdot H_f(\mathbf{x}_0)\mathbf{h}$$

   where $\mathbf{h} = \mathbf{x} - \mathbf{x}_0 = (x - x_0, y - y_0)$ and H_f is the "Hessian" matrix

$$H_f = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix}.$$

   Of course, since we're asking f to be twice continuously differentiable in this case, $f_{xy} = f_{yx}$ by Clairaut's theorem and so this matrix is "symmetric", i.e., $H_f = H_f^\top$.

7. You should know little-o and big-o notation. In particular, if we say that $f(\mathbf{h}) = o(\|\mathbf{h}\|)$ as $\mathbf{h} \to \mathbf{0}$, we mean that

$$\lim_{\mathbf{h}\to 0} \frac{f(\mathbf{h})}{\|\mathbf{h}\|} = 0.$$

   Equivalently, for each $\epsilon > 0$ there is a $\delta > 0$ for which

$$|f(\mathbf{h})| < \epsilon\|\mathbf{h}\|$$

   whenever $\|\mathbf{h}\| < \delta$. We also say that $g(\mathbf{h}) = O(\mathbf{h})$ as $\mathbf{h} \to 0$ if there exist positive numbers δ and M for which

$$|f(\mathbf{h})| \leq M\|\mathbf{h}\|$$

   whenever $\|\mathbf{h}\| < \delta$.

8. For a $d \times d$ symmetric matrix A, we say that A is positive definite if its quadratic form Q_A has the property that

$$Q_A(\mathbf{h}) = \mathbf{h} \cdot A\mathbf{h} \geq 0$$

   for all $\mathbf{h} \in \mathbb{R}^d$ and only takes the value 0 when $\mathbf{h} = \mathbf{0}$. Similarly, A (and Q_A) are negative-definite if $Q_A(\mathbf{h}) \leq 0$ for all $\mathbf{h} \in \mathbb{R}^d$ and has the value 0 only when $\mathbf{h}0 = 0\mathbf{0}$.

9. Given a function $f : \mathcal{D} \subseteq \mathbb{R}^d \to \mathbb{R}$, you should know what it means for $\mathbf{x}_0 \in \mathcal{D}$ to be a:
   (a) local maximum
   (b) local minimum
   (c) saddle point
   (d) global maximum (on $\mathcal{D}$)
   (e) global maximum (on $\mathcal{D}$)
   (f) critical point

10. In $\mathbb{R}^d$, you should know the definitions of the following (all appeared in Homework 7):
   (a) Open set.
   (b) Closed set.
   (c) Closure $\overline{E}$ of a set $E \subseteq \mathbb{R}^d$.
   (d) Interior $\mathrm{Int}(E)$ of a set $E \subseteq \mathbb{R}^d$.
   (e) Boundary ∂E of a set $E \subseteq \mathbb{R}^d$.

11. Know what it means for $\mathcal{G}$ to be a grid for a rectangle $R \subseteq \mathbb{R}^d$.

12. For a subset $E \subseteq R \subseteq \mathbb{R}^d$ and a grid $\mathcal{G}$ of R, know how the outer and inner Jordan sums $V(E;\mathcal{G})$ and $v(E;\mathcal{G})$ are defined.

13. You should know the definitions of the inner and outer Jordan volume: $\overline{\text{Vol}}(E)$ and $\underline{\text{Vol}}(E)$.

14. You should know what it means for a bounded set $E \subseteq \mathbb{R}^d$ to be a Jordan region (and so have a Jordan volume). Also, you should know what it means for E to have volume zero.

15. Given a Jordan region $E \subseteq \mathbb{R}^d$ and a bounded function $f : E \to \mathbb{R}$, you should be able to define (and compute in many cases) the following objects:

    (a) For a grid $\mathcal{G}$ (of a rectangle $R \supseteq E$), $U(f,\mathcal{G})$.

    (b) For a grid $\mathcal{G}$, $L(f,\mathcal{G})$.

    (c) The upper integral
    $$\overline{\int_E} f(\mathbf{x})\,d\mathbf{x}$$

    (d) The lower integral
    $$\underline{\int_E} f(\mathbf{x})\,d\mathbf{x}.$$

    Note that Wade writes $(U)\int_E$ for the former and $(L)\int_E$ for the latter.

    (e) The integral:
    $$\int_E f = \int_E f(\mathbf{x})\,d\mathbf{x} = \int_E f\,dV$$
    and, in the case that $d = 2$, dV might be replaced by dA.

16. Iterated integrals.

17. Projectable regions

Results (Theorems, propositions, lemmas, corollaries):

For the following results, unless otherwise mentioned, you should know the statement of the result precisely and have a really good idea of how they are proved – ideally, you should be able to reproduce the proof.

1. You should know the proof of the chain rule (Theorem 12.23) and its proof (either from Wade or from class.

2. You should know how the chain rule allows you to write directional derivaitves. Specifically, you should be able to prove the following: If $f : \mathcal{D} \subseteq \mathbb{R}^d \to \mathbb{R}$ is differentiable at $\mathbf{a} \in \mathcal{D}$, then, for any direction vector $\mathbf{v}$, we have
$$D_{\mathbf{v}} f(\mathbf{a}) = \nabla f(\mathbf{a}) \cdot \mathbf{v}$$

3. You should know Theorem 11.27 and its proof. Note: It has a nice simplification for real-valued functions. Can you prove this?

4. You should know Corollaries 11.29 and 11.30.

5. You should know Clairaut's theorem (and have an idea of its proof). You should also know the examples in Wade and your homework which show that, in the case second-order partials are not continuous, mixed partials need not be equal.

6. You should know Taylor's theorem with remainder given by the MVT. Here is the statement I gave in class:

Theorem 1. *Let $f : \mathcal{D} \subseteq \mathbb{R}^d \to \mathbb{R}$ and let $\mathcal{O} \subseteq \mathcal{D}$ be an open convex region. Suppose that $f \in C^{(n+1)}(\mathcal{O})$ (that is, for each $\alpha \in \mathbb{N}^d$ for which $|\alpha| \leq n+1$, $D^\alpha f$ exists and is continuous on $\mathcal{O}$.) Finally, take $\mathbf{x}_0 \in \mathcal{O}$. Then, for any $\mathbf{x} \in \mathcal{O}$, there is a $\mathbf{c}$ on the line segment between $\mathbf{x}$ and $\mathbf{x}_0$ for which*

$$f(\mathbf{x}) = \sum_{|\alpha| \leq n} \frac{D^\alpha(\mathbf{x}_0)}{\alpha!}(\mathbf{x} - \mathbf{x}_0)^\alpha + \sum_{|\alpha|=n+1} \frac{D^\alpha(\mathbf{c})}{\alpha!}(\mathbf{x} - \mathbf{x}_0)^\alpha.$$

You certainly don't need to know how to prove this theorem (the proof is actually just a careful application of the MVT, just like we did in one dimension).

7. You should know the following corollary of Taylor's theorem that's helpful.

   Corollary 2. *Let $f \in C^{(n+1)}(\mathcal{O})$ for an open region $\mathcal{O} \subseteq \mathbb{R}^d$. Then, for any $\mathbf{x}_0 \in \mathcal{O}$ and compact and convex set $\mathbf{x}_0 \in K \subseteq \mathcal{O}$,*

   $$f(\mathbf{x}) = \sum_{|\alpha| \leq n} \frac{D^\alpha(\mathbf{x}_0)}{\alpha!}(\mathbf{x} - \mathbf{x}_0)^\alpha + o(\|\mathbf{x} - \mathbf{x}_0\|^n)$$

   for $\mathbf{x} \in K$ where the little-o notation is in reference to $\mathbf{h} = \mathbf{x} - \mathbf{x}_0 \to \mathbf{0}$.

   In fact, you should try to prove this corollary using Taylor's theorem.

8. Of course, as with our previous discussion of the Hessian, we obtain the following special case of the corollary:

   Corollary 3. *Let $f \in C^3(\mathcal{O})$. Then, for any $\mathbf{x}_0 \in \mathcal{O}$ and $\mathbf{x}_0 \in K \subseteq \mathcal{O}$ where K is compact and convex, then*

   $$f(\mathbf{x}) = f(\mathbf{x}_0) + \nabla f(\mathbf{x}_0) \cdot \mathbf{h} + \frac{1}{2}\mathbf{h} \cdot H_f(\mathbf{x}_0)\mathbf{h} + o(\|\mathbf{h}\|^2)$$

   as $\mathbf{h} = (\mathbf{x} - \mathbf{x}_0) \to \mathbf{0}$. Here; H_f is the (necessarily symmetric) Hessian matrix.

9. You should know the first derivative test: If $f \in C^1(\mathcal{O})$ for an open set $\mathcal{O} \subseteq \mathbb{R}^d$ has a local maximum or minimum at $\mathbf{x}_0 \in \mathcal{O}$, then $\nabla f(\mathbf{x}_0) = \mathbf{0}$.

10. You should also know the second derivative test:

    Proposition 4. *Let $f \in C^2(\mathcal{O})$ for an open set $\mathcal{O}$ and suppose that f has a critical point at $\mathbf{x}_0 \in \mathcal{O}$ with $\nabla f(\mathbf{x}_0) = \mathbf{0}$ and denote by $H = H_f(\mathbf{x}_0)$ the Hessian at $\mathbf{x}_0$.*

    (a) If the quadratic form Q_H is positive definite, then f has a local minimum at $\mathbf{x}_0$.

    (b) If the quadratic form Q_H is negative definite, then f has a local maximum at $\mathbf{x}_0$.

    (c) If the quadratic form $Q_H(\mathbf{h})$ takes on both positive and negative values for $\mathbf{h}$ close to $\mathbf{0}$, then $\mathbf{x}_0$ is a saddle point.

    We discussed the proof of this in class for C^3 functions. A nice proof for C^2 functions appears in the textbook in Theorem 11.56. Of course, knowing when a quadratic form/matrix are positive definite really requires some linear algebra (for $d \geq 3$). Thus, on an exam, I would give you something that would make this very clear.

11. In the case that $d = 2$, you should know the special case of the above theorem appearing as Theorem 11.59 in Wade.

12. You should know all of the results we discussed on equivalence relations and appearing in Chapter 7 of "Introduction to Mathematics" by Scott A. Taylor.

13. You should know the basic topological results appearing in Homework 7.

14. Here are the results you must know (and their proofs) in Chapter 12:

(a) Remark 12.1
(b) Remark 12.2
(c) Remark 12.3
(d) Remark 12.6
(e) Remark 12.7
(f) Remark 12.9
(g) Remark 12.11
(h) Theorem 12.12
(i) Remark 12.13
(j) Remark 12.16
(k) Remark 12.17
(l) Remark 12.19
(m) Theorem 12.20
(n) Theorem 12.21
(o) Theorem 12.22
(p) Theorem 12.23 (You don't need to know how to prove this, just have an idea why it's valid)
(q) Theorem 12.25
(r) Theorem 12.26 (and it's various corollaries we proved in class).
(s) Lemma 12.30 (There is no need to be able to prove this, but you should have a good idea of how it's proved). Same for Lemma 12.36
(t) Theorem 12.31
(u) Theorem 12.39
(v) On our last day of class, I will prove/(discuss the proof of) the change of variable formula appearing as Theorem 12.45 in Wade. Truthfully, I'm not crazy about Wade's treatment and so I will instead discuss the corresponding result from Spivak's "Calculus on Manifolds". I emailed you all a link/pointer to the relevant section in Spivak. Overall, you should know the change of variables formula, how to use it, but you don't need to know the proof more then the key idea that the absolute value of the determinant of a matrix tells you how the volume of parallelepipeds scale under matrix transformation.

0.1 Things you should be able to do:

1. Use the chain rule. Both the matrix and scalar versions.
2. Determine when partial derivatives (some of order 2 or higher) exist.
3. Apply Taylor's formula to estimate things as you did in Homework 5.
4. You should be able to prove that a given relation is an equivalence relation. You should be able to describe the equivalence classes for an equivalence relation. You should also be able to check in an operation is well-defined relative to an equivalence relation.
5. You should be able to find the interior, closure, boundary of relatively simple set in $\mathbb{R}^d$ and justify your claims.
6. For reasonable simple bounded sets $E \subseteq \mathbb{R}^d$, you should be able to compute $V(E;\mathcal{G})$, $v(E;\mathcal{G})$, and their inner and outer volumes.
7. For relatively simple sets, you should be able to determine their Jordan volume and justify your assertions. For example, could you prove that the line segment $\{(x,x) : x \in [0,1]\}$ has zero volume in $\mathbb{R}^2$?

8. For relatively simply functions f on relatively simple Jordan regions $E \subseteq \mathbb{R}^d$, you should be able to compute upper and lower sums and their integrals, if they exist.
9. You should be able to apply Fubini's theorem.
10. You should be able to apply Fubini's theorem for projectable regions.
11. You should be able to calculate the volume/areas of things using Fubini's theorem.
12. You should be able to use change-of-variables (and maybe some Fubini's theorem) to compute lots of volumes and integrals.
13. In other words, for the above items, you should be able to do many computations like you did in the final homework assignment.

Things not to worry about:

You know about the following things and why they are important (at least for us) but I won't ask you anything detailed about them.

1. If it wasn't listed, just email me and ask.