Math 165: Homework 7

Please complete the following exercises below and write up your solutions consistent with the directions in the syllabus. Your solutions are due on Thursday, May 1st at 10:00AM in the appropriate box outside my office door. If you get stuck on any part of the homework, please come and see me. More importantly, have fun!

In what follows, we recall that, for $\mathbf{x} \in \mathbb{R}^d$ and $R > 0$, the open ball with center $\mathbf{x}$ and radius R is the set

$$B_R(\mathbf{x}) = \{\mathbf{y} \in \mathbb{R}^d : \|\mathbf{y} - \mathbf{x}\| < R\}.$$

A set $\mathcal{O} \subseteq \mathbb{R}^d$ is said to be *open* if, for every $\mathbf{x} \in \mathcal{O}$, there is some $r > 0$ for which

$$\mathbf{x} \in B_r(\mathbf{x}) \subseteq \mathcal{O}.$$

For a set $E \subseteq \mathbb{R}^d$, a point $\mathbf{x}$ is said to be an *accumulation (or limit) point* of E if, for every $r > 0$, the ball $B_r(\mathbf{x})$ contains a point of E that is distinct from $\mathbf{x}$. We remark that this definition does not require $\mathbf{x}$ to be a member of E. A set $F \subseteq \mathbb{R}^d$ is said to be *closed* if it contains all of its limit points.

Exercise 1. Please prove the following statements:

1. For any $\mathbf{x} \in \mathbb{R}^d$ and $R > 0$, the open ball $B_R(\mathbf{x})$ is open.
2. A set F is closed if and only if its complement $F^c = \mathbb{R}^d \setminus F$ is open.
3. If $\{\mathcal{O}_\alpha\}$ is a, possibly infinite, collection of open sets, then
$$\bigcup_\alpha \mathcal{O}_\alpha$$
is open.
4. If $\mathcal{O}_1, \mathcal{O}_2, \dots, \mathcal{O}_N$ is a finite list of open sets, then
$$\bigcap_{n=1}^N \mathcal{O}_n$$
is an open set. Is true if this list were infinite?
5. If $\{F_\alpha\}$ is a, possibly infinite, collection of closed sets, then
$$\bigcap_\alpha F_\alpha$$
is a closed set.
6. A set F is closed if and only if it has the following property: If $\{\mathbf{x}_n\} \subseteq F$ is a convergent sequence of points all of which belong to F, then $\lim_{n\to\infty} \mathbf{x}_n \in F$.

Exercise 2. Given a set $E \subseteq \mathbb{R}^d$, we define its *closure and interior*, respectively, by

$$\overline{E} = \bigcap_{\substack{F \supseteq E \\ F \text{ closed}}} F \qquad \text{and} \qquad \operatorname{int}(E) = \bigcup_{\substack{\mathcal{O} \subseteq E \\ \mathcal{O} \text{ open}}} \mathcal{O}.$$

Note that Wade writes $\overset{\circ}{E}$ for $\operatorname{int}(E)$. With these sets in hand, we define the *boundary of* E to be the set

$$\partial E = \overline{E} \setminus \operatorname{int}(E).$$

Prove the following statements:

1. For any set $E \subseteq \mathbb{R}^d$, $\overline{E}$ is closed and $\text{int}(E)$ is open. Also, explain why this means that $\overline{E}$ is the smallest closed set containing E and that $\text{int}(E)$ is the largest open set contained by E.

2. $E = \overline{E}$ if and only if E is closed.

3. $E = \text{int}(E)$ if and only if E is open.

4. For any set $E \subseteq \mathbb{R}^d$,
$$\overline{E} = E \cup \text{Acc}(E)$$
where $\text{Acc}(E) = \{\mathbf{x} \in \mathbb{R}^d : \mathbf{x} \text{ is an accumulation point of } E\}$..

Exercise 3. We recall, a set $\mathcal{D} \subseteq \mathbb{R}^d$ is called disconnected if there exist disjoint open sets $\mathcal{O}$ and $\mathcal{U}$ for which $\mathcal{D} \cap \mathcal{O} \neq \varnothing$, $\mathcal{D} \cap \mathcal{U} \neq \varnothing$, and
$$\mathcal{D} \subseteq \mathcal{O} \cup \mathcal{U}.$$

A set $\mathcal{C} \subseteq \mathbb{R}^d$ is called *connected* if it is not disconnected. Let's also recall that a set $\mathcal{P} \subseteq \mathbb{R}^d$ is called *path connected* if, for all $\mathbf{x}, \mathbf{y} \in \mathcal{P}$, there is a continuous function $\mathbf{r} : [0,1] \to \mathcal{P}$ for which $\mathbf{r}(0) = \mathbf{x}$ and $\mathbf{r}(1) = \mathbf{y}$.

1. Let $\{a_1, b_1, a_2, b_2, \ldots, a_d, b_d\}$ be a set of real numbers for which $a_k < b_k$ for $k = 1, 2, \ldots, d$. Prove that the rectangle
$$R = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_d, b_d] = \{\mathbf{x} = (x_1, x_2, \ldots, x_d) \in \mathbb{R}^d : a_k \leq x_k \leq b_k \text{ for all } k = 1, 2, \ldots, d\}$$
is path connected.

2. Prove: If $E \subseteq \mathbb{R}^d$ is path connected, then it is connected. *Hint: Let $\mathcal{U}$ and $\mathcal{O}$ be disjoint open sets for which $E \cap \mathcal{O} \neq \varnothing$ and $E \cap \mathcal{U} \neq \varnothing$. Since E is path connected, we may take a continuous path $\mathbf{r} : [0,1] \to E$ for which $\mathbf{r}(0) = \mathbf{x} \in E \cap \mathcal{O}$ and $\mathbf{r}(1) = \mathbf{y} \in E \cap \mathcal{U}$. Set*
$$t_o = \sup\{t \in [0,1] : \mathbf{r}(t) \in \mathcal{O}\} \qquad \textit{and} \qquad t_u = \inf\{t \in [0,1] : \mathbf{r}(t) \in \mathcal{U}\}.$$
Use these (and the continuity of $\mathbf{r}$) to show that, for $t_o \leq t \leq t_u$, $\mathbf{r}(t) \in E$ but not in $\mathcal{O}$ or $\mathcal{U}$. Hence $E \not\subseteq \mathcal{O} \cap \mathcal{U}$.

3. Let $E \subseteq \mathbb{R}^d$ and let $R \subseteq \mathbb{R}^d$ be a rectangle for which $R \cap \overline{E} \neq \varnothing$ and $R \not\subseteq \text{int}(E)$. Prove that $R \cap \partial E \neq \varnothing$. *Hint: First prove $\mathbb{R}^d = (\overline{E})^c \cup \partial E \cup \text{int}(E)$.*

Exercise 4. Please do the following exercises:

1. Exercise 1 in Section 12.1 of Wade, 2nd edition.
2. Exercise 2 in Section 12.1 of Wade, 2nd edition.
3. Exercise 5 in Section 12.1 of Wade, 2nd edition.

Exercise 5. Please do the following exercises:

1. Exercise 1 in Section 12.2 of Wade.
2. Exercise 2 in Section 12.2 of Wade.
3. Exercise 4 in Section 12.2 of Wade.

Exercise 6. Please do the following exercises:

1. Exercise 5 in Section 12.2 of Wade.
2. Exercise 6 in Section 12.2 of Wade.
3. Exercise 8 in Section 12.2 of Wade.