Math 165: Homework 5

Please complete the following exercises below and write up your solutions consistent with the directions in the syllabus. Your solutions are due on Thursday, April 17th at 10:00AM in the appropriate box outside my office door. If you get stuck on any part of the homework, please come and see me. More importantly, have fun!

Exercise 1.

1. Let A be an $n \times n$ matrix. Prove that
$$\|A\mathbf{x}\| \leq \|A\|\|\mathbf{x}\|$$
for all $\mathbf{x} \in \mathbb{R}^n$ where $\|A\|$ denotes the matrix norm of A, i.e.,
$$\|A\| = \sqrt{\|\mathbf{a}_1\|^2 + \|\mathbf{a}_2\|^2 + \cdots + \|\mathbf{a}_n\|^2}$$
where, for each $k = 1, 2, \dots, n$, $\mathbf{a}_k$ denotes the kth row of A.

2. Use the above and the Cauchy-Schwarz inequality to show that the associated "quadratic form" $Q_A : \mathbb{R}^n \to \mathbb{R}$ given by
$$Q_A(\mathbf{x}) = \mathbf{x} \cdot A\mathbf{x}$$
for $\mathbf{x} \in \mathbb{R}^n$ satisfies
$$|Q_A(\mathbf{x})| \leq \|A\|\|\mathbf{x}\|^2$$
for all $\mathbf{x} \in \mathbb{R}^n$.

3. We shall use the result above to quantify the approximation yielded by affine approximation/Taylor's theorem for scalar-valued functions of two variables. To this end, prove the following proposition:

   Proposition A. *Let $\mathcal{D} \subseteq \mathbb{R}^2$ be a non-empty open set and let $f : \mathcal{D} \to \mathbb{R}$ be twice continuously differentiable, i.e., $f \in C^2(\mathcal{D})$. If $K \subseteq \mathcal{D}$ is compact, then there is a number $M > 0$ for which*
$$\left| f_{xx}(\mathbf{x})h_1^2 + 2f_{xy}(\mathbf{x})h_1h_2 + f_{yy}(\mathbf{x})h_2^2 \right| \leq M\|\mathbf{h}\|^2$$
for all $\mathbf{x} \in K$ and $\mathbf{h} = (h_1, h_2) \in \mathbb{R}^2$.

   Hint: You can write the term in absolute values as the quadratic form $Q_{D^2f(\mathbf{x})}(\mathbf{h})$ where
$$D^2f(\mathbf{x}) = \begin{pmatrix} f_{xx}(\mathbf{x}) & f_{xy}(\mathbf{x}) \\ f_{yx}(\mathbf{x}) & f_{yy}(\mathbf{x}) \end{pmatrix}$$
for $\mathbf{x} \in \mathcal{D}$ (in view of Clairaut's theorem). Thus, if you can find a 'global' bound on the matrix norm $\|D^2f(\mathbf{x})\|$ as $\mathbf{x}$ ranges over K, you can simply use what you proved in the previous item.

4. Prove the following theorem (using Taylor's theorem and what you've proven in the previous item):

   Theorem B. *Let $f \in C^2(\mathcal{D})$ where $\mathcal{D}$ is a non-empty open subset of $\mathbb{R}^2$ and, given $\mathbf{x}_0 \in \mathcal{D}$, consider the affine approximation of f at $\mathbf{x}_0$ defined by*
$$L_{f,\mathbf{x}_0}(\mathbf{x}) = f(\mathbf{x}_0) + \boldsymbol{\nabla} f(\mathbf{x}_0) \cdot (\mathbf{x} - \mathbf{x}_0)$$
for $\mathbf{x} \in \mathbb{R}^2$. Then, for any convex and compact set K for which $\mathbf{x}_0 \in K \subseteq \mathcal{D}$, there is a constant $M > 0$ for which
$$|f(\mathbf{x}) - L_{f,\mathbf{x}_0}(\mathbf{x})| \leq \frac{M}{2}\|\mathbf{x} - \mathbf{x}_0\|^2$$
for all $\mathbf{x} \in K$.

5. Generalize the above theorem to scalar-valued functions f on $\mathcal{D} \subseteq \mathbb{R}^n$. You need to state a theorem precisely and, while you don't need to give a full proof, you should give me a really good idea how one should go about the proof.

Exercise 2. In this exercise, we shall use the previous exercise to estimate some number by hand and also understand the accuracy of our estimates. Here are three numbers:

1. $$e^{0.1}\cos(-0.01)$$
2. $$\sqrt{(1.01)^2 + (2.01)^3 - (0.001)}$$
3. $$\big(\cos\big((0.1)^2\big)\big)^{2.01}$$

For each of the above numbers, do the following:

1. Identify a sufficiently differentiable scalar-valued function f and a point $\mathbf{x} \in \mathbb{R}^n$ for which the number is exactly $f(\mathbf{x})$. Also, identify a nearby point $\mathbf{x}_0$ for which $f(\mathbf{x}_0)$ is extremely easy to compute (by hand).
2. Using the f, $\mathbf{x}$ and $\mathbf{x}_0$ as above, estimate $f(\mathbf{x})$ by computing (by hand) $L_{f,\mathbf{x}_0}(\mathbf{x})$.
3. Quantify the error. Using the previous exercise, find a constant M for which
$$|f(\mathbf{x}) - L_{f,\mathbf{x}_0}(\mathbf{x})| \leq \frac{M}{2}\|\mathbf{x} - \mathbf{x}_0\|^2.$$
Determine the right hand side of this inequality (to get an upper bound on how far your estimate is from the true number).
4. Now, ask your calculator (or computational software) to compute the number. Does the error you got in the previous item make sense? Explain.

Exercise 3. Prove that the function
$$f(x,y) = \begin{cases} xy\frac{x^2-y^2}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$
does not satisfy the conclusion of Clairaut's theorem at $\mathbf{x}_0 = (0,0)$. Does it satisfy the conclusion of the theorem at $\mathbf{x}_0 = (1,1)$? Explain.

Exercise 4. Let $\mathcal{H}$ be the hyperboloid of one sheet given by $x^2 + y^2 - z^2 = 1$.

1. Prove that, at every point $(a,b,c) \in \mathcal{H}$, $\mathcal{H}$ has a tangent plane whose normal vector is given by $(-a,-b,c)$.
2. Find an equation of each plane tangent to $\mathcal{H}$ which is perpendicular to the xy plane.
3. Find an equation of each plane tangent to $\mathcal{H}$ which is parallel to the plane $x + y - z = 1$.

Exercise 5. Find and classify (using the first and second derivative tests) all critical points of the functions:

1. $f(x,y) = x^2 - xy + y^3 - y$
2. $f(x,y) = \sin(x) + \cos(y)$
3. $f(x,y,z) = e^{x+y}\cos(z)$

Exercise 6. For each of the following, find the maximum and minium of f on H.

1. $f(x,y) = x^2 + 2x - y^2$ and $H = \{(x,y) : x^2 + 4y^2 \leq 4\}$.
2. $f(x,y) = x^2 + 2xy + 3y^2$ and H is the region bounded by the triangle with vertices $(1,0)$, $(1,2)$ and $(3,0)$.
3. $f(x,y) = x^3 + 3xy - y^3$ and $H = [-1,1]^2$.