Math 165: Homework 4

Please complete the following exercises below and write up your solutions consistent with the directions in the syllabus. Your solutions are due on Thursday, March 20th at 10:00AM in the appropriate box outside my office door. If you get stuck on any part of the homework, please come and see me. More importantly, have fun!

Exercise 1. In this exercise, you will prove that sequences in $\mathbb{R}^d$ are Cauchy if and only if they are convergent. You will do this in a different way than our textbook suggests[1].

1. First, prove that, for each $k = 1, 2, \ldots, d$,

$$|y_k| \leq \|\mathbf{y}\| \leq \sqrt{d}\left(\max_{j=1,2,\ldots,d} |y_j|\right)$$

for all $\mathbf{y} = (y_1, y_2, \ldots, y_d) \in \mathbb{R}^d$; here, as usual, $\|\cdot\|$ denotes the Euclidean norm on $\mathbb{R}^d$.

2. By applying the inequality above and directly invoking the theorem that says sequences of real numbers are convergent if and only if they are Cauchy, prove the following $\mathbb{R}^d$ analogue:

Theorem A. *Let $\{\mathbf{y}_n\}$ be a sequence of points/vectors in $\mathbb{R}^d$. Then $\{\mathbf{y}_n\}$ is convergent if and only if it is Cauchy.*

Exercise 2. Determine if the following sequences converge. Prove your assertions.

1.
$$\mathbf{x}_n = \left(\frac{1}{n}, 1 - \frac{1}{n^2}\right) \in \mathbb{R}^2.$$

2.
$$\mathbf{y}_n = \left(\frac{k}{k+1}, \sin(1/k)\right) \in \mathbb{R}^2$$

3.
$$\mathbf{x}_k = \left(k - \sqrt{k^2 + k}, k^{1/k}, \frac{1}{k}\right) \in \mathbb{R}^3$$

Exercise 3. In class, we discussed the following theorem:

Theorem B. *Let $F : \mathbb{R}^n \to \mathbb{R}^m$ and let $\mathbf{x}_0$ be a limit point of the domain of F,* Dom(F)*. Then*

$$\lim_{\mathbf{x} \to \mathbf{x}_0} F(\mathbf{x}) = \mathbf{y}_0$$

if and only if, for every sequence $\{\mathbf{x}_n\} \subseteq \mathrm{Dom}(F)$ such that $\mathbf{x}_n \to \mathbf{x_0}$, we have $F(\mathbf{x}_n) \to \mathbf{y}_0$.

1. Prove the theorem.

2. Use the theorem to conclude that the limit

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2 + y^2}$$

does not exist.

Exercise 4. Determine whether or not the following functions are continuous at the points indicated. Using δ-ϵ arguments (or the theorem in the previous exercise), prove your assertions.

[1] The proof that Wade suggests if via Bolzano-Weierstrass. In this exercise, you will show that simply using the theorem's analogue in $\mathbb{R}$ and one inequality if enough.

1. $f : \mathbb{R}^2 \to \mathbb{R}$ defined by
$$f(x,y) = \begin{cases} \sin\left(\frac{1}{x^2+y^2}\right) & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$
at the point $(x,y) = (0,0)$.

2. For any $\alpha > 0$, the function $g : \mathbb{R}^2 \to \mathbb{R}$ defined by
$$g(x,y) = \begin{cases} \frac{x^\alpha y^4}{x^2+y^4} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$
at the point $(x,y) = (0,0)$.

3. $F : \mathbb{R} \to \mathbb{R}^2$ defined by
$$F(t) = \begin{pmatrix} |t| \\ t\sin\big(1/t^2\big) \end{pmatrix}$$
for $t \in \mathbb{R}$ at the point $t = 0$; here we take $t\sin\big(1/t^2\big)$ to take the value 0 at $t = 0$.

Exercise 5. We first recall the definition:

Definition C. Let $\mathcal{D} \subseteq \mathbb{R}^d$. We say that $\mathcal{D}$ is "path connected" if, for any $\mathbf{a}, \mathbf{b} \in \mathcal{D}$, there is a continuous map $\mathbf{r} : [0,1] \to \mathcal{D}$ such that $\mathbf{r}(0) = \mathbf{a}$ and $\mathbf{r}(1) = \mathbf{b}$.

1. Prove that $\mathbb{R}^d$ is path connected.

2. Determine if the set consisting the y axis and the line $x = 1$ in $\mathbb{R}^2$, i.e.,
$$\mathcal{D} = \{(x,y) \in \mathbb{R}^2 : x = 0 \text{ or } x = 1\}$$
is path connected. Prove your assertion.

3. Prove the result:

Theorem D. *Let $f : \mathcal{D} \to \mathbb{R}$ be continuous. If $\mathcal{D}$ is path connected, then, for any $\mathbf{a}, \mathbf{b} \in \mathbb{R}^d$, and any c between the real numbers $f(\mathbf{a})$, and $f(\mathbf{b})$, there is some $\mathbf{c} \in \mathcal{D}$ such that $f(\mathbf{c}) = c$.*

4. Let's now consider vector valued functions. Show that the above result is false if $f : \mathbb{R}^2 \to \mathbb{R}^2$ continuous and we regard "in between" to be anywhere along the line between $f(\mathbf{a})$ and $f(\mathbf{b})$ in $\mathbb{R}^2$.

This is the end of the homework.