This is the ninth and final homework assignment for Math 160 and it is broken into two parts. The first part of the homework consists of exercises you should do (and I'll expect you to do) but you needn't turn in. As these exercises will not be graded, if you would like help with them or just want to make sure you're doing them correctly, you should (always) feel free to come to office hours (or the nightly TA sessions). The second part is the part you are expected to turn in. More precisely, please complete all problems in Part 2, write up clear and thorough solutions for them (consistent with the directions given in the syllabus) and hand them in. Your write-ups are due on **Thursday, December 4th** in the box outside my office door. As always, please come and see me early if you get stuck on any part of this assignment. I am here to help!

Part 1 (Do not turn in)

Exercise 1. Do these exercises to practice double integrals over rectangles.

a. Let $\mathcal{R} = [0,4] \times [-1,2]$. Estimate the value of

$$\iint_{\mathcal{R}} (1 - xy^2)\, \mathrm{d}A$$

using a Riemann sum with $\Delta x = 2$ and $\Delta y = 1$ and sampling points in the...

(i) ...lower right corners.

(ii) ...upper left corners.

b. Evaluate each double integral by first identifying it as the volume of a solid.

(i) $\iint_{\mathcal{R}} 3\, \mathrm{d}A$, where $\mathcal{R} = [-2,2] \times [1,6]$

(ii) $\iint_{\mathcal{R}} (5 - x)\, \mathrm{d}A$, where $\mathcal{R} = [0,5] \times [0,3]$

(iii) $\iint_{\mathcal{R}} (4 - 2y)\, \mathrm{d}A$, where $\mathcal{R} = [0,1] \times [0,1]$

Exercise 2. Do these exercises to practice iterated integrals over rectangles.

a. Calculate each iterated integral below.

(i) $\displaystyle\int_1^4 \int_0^2 \left(6x^2y - 2x\right)\, \mathrm{d}y\, \mathrm{d}x$

(ii) $\displaystyle\int_0^2 \int_0^4 y^3 e^{2x}\, \mathrm{d}y\, \mathrm{d}x$

(iii) $\displaystyle\int_{-3}^3 \int_0^{\pi/2} \left(y + y^2\cos(x)\right)\, \mathrm{d}x\, \mathrm{d}y$

b. Calculate each double integral below.

(i) $\iint_{\mathcal{R}} \sin(x - y)\, \mathrm{d}A$, where $\mathcal{R} = [0, \frac{\pi}{2}] \times [0, \frac{\pi}{2}]$

(ii) $\displaystyle\iint_{\mathcal{R}} \frac{xy^2}{x^2+1}\, \mathrm{d}A$, where $\mathcal{R} = [0,1] \times [-3,3]$

(iii) $\displaystyle\iint_{\mathcal{R}} \frac{x}{1+xy}\, \mathrm{d}A$, where $\mathcal{R} = [0,1] \times [0,1]$

c. Find the volume of each solid below.

(i) The solid that lies below the plane $4x + 6y - 2z + 15 = 0$ and above the rectangle $\mathcal{R} = [-1,2] \times [-1,1]$

(ii) The solid that lies below the hyperbolic paraboloid $z = 3y^2 - x^2 + 2$ and above the rectangle $\mathcal{R} = [-1,1] \times [1,2]$

(iii) The solid enclosed by the surface $z = 1 + e^x \sin(y)$ and the planes $x = \pm 1$, $y = 0$, $y = \pi$, and $z = 0$.

Exercise 3. Do these exercises to practice double integrals over general regions.

a. Evaluate each iterated integral below.

(i) $\displaystyle\int_0^4 \int_0^{\sqrt{y}} xy^2 \, \mathrm{d}x \, \mathrm{d}y$

(ii) $\displaystyle\int_0^1 \int_{x^2}^{x} (1 + 2y) \, \mathrm{d}y \, \mathrm{d}x$

(iii) $\displaystyle\int_0^2 \int_y^{2y} 2y \, \mathrm{d}x \, \mathrm{d}y$

b. Evaluate each double integral below.

(i) $\displaystyle\iint_{\mathcal{R}} y^2 \, \mathrm{d}A$, where $\mathcal{R}$ is the region where $-1 \le y \le 1$ and $(-y-2) \le x \le y$

(ii) $\displaystyle\iint_{\mathcal{R}} \frac{y}{x^5+1} \, \mathrm{d}A$, where $\mathcal{R}$ is the region where $0 \le x \le 1$ and $0 \le y \le x^2$

(iii) $\displaystyle\iint_{\mathcal{R}} x \, \mathrm{d}A$, where $\mathcal{R}$ is the region where $0 \le x \le \pi$ and $0 \le y \le \sin(x)$

c. The terminology for 2D regions is all over the place; y-simple regions are also called "vertically simple" or "Type I". Similarly, x-simple regions are also called "horizontally simple" or "Type II". Sketch an example of a region that is...

(i) ... y-simple but not x-simple.

(ii) ... x-simple but not y-simple.

(iii) ... both x-simple and y-simple.

(iv) ... neither x-simple nor y-simple.

d. Evaluate each double integral below.

(i) $\displaystyle\iint_{\mathcal{R}} x\cos(y) \, \mathrm{d}A$, where $\mathcal{R}$ is the region bounded by $y = 0$, $y = x^2$, and $x = 1$.

(ii) $\displaystyle\iint_{\mathcal{R}} (x^2 + 2y) \, \mathrm{d}A$, where $\mathcal{R}$ is the region in the first quadrant bounded by $y = x$ and $y = x^3$.

(iii) $\displaystyle\iint_{\mathcal{R}} y^2 \, \mathrm{d}A$, where $\mathcal{R}$ is the triangle with vertices $(0,1)$, $(1,2)$, and $(4,1)$.

(iv) $\displaystyle\iint_{\mathcal{R}} xy^2 \, \mathrm{d}A$, where $\mathcal{R}$ is the region enclosed by the graphs of $x = 0$ and $x = \sqrt{1 - y^2}$.

e. Find the volume of each solid below.

(i) The solid under the plane $x - 2y + z = 1$ and above the region in the xy-plane bounded by $x + y = 1$ and $x^2 + y = 1$.

(ii) The solid bounded by the coordinate planes and the plane $3x + 2y + z = 6$.

(iii) The solid bounded by the planes $z = x$, $y = x$, $x + y = 2$, and $z = 0$.

Part 2 (Solutions for these problems are due in the appropriate box outside my office door at 11:00AM on December 4th)

Problem 1. While we mostly discussed optimization of two-variable functions in class, some of our techniques generalize to three (and more) variables. A *critical point* of a function $f(x_1, x_2, \ldots, x_n)$ of n variables is a point $(x_1, x_2, \ldots, x_n)$ such that either $\nabla f(x_1, x_2, \ldots, x_n) = \langle 0, 0, \ldots, 0 \rangle$ or where f fails to be differentiable. It is also true in general that local extrema of $f(x_1, x_2, \ldots, x_n)$ must occur at critical points.

a. Find all critical points of $f(x, y, z) = xyz + z^2 + xy - x - y - z$. Are any of your critical points global minima or maxima?

b. Find all critical points of $g(x, y, z) = 4x^2 + y^2 + 18z^2 - 2xy - 6x + 9$. Are any of your critical points global minima or maxima?

   (Hint: g can also be written as $g(x, y, z) = (x - y)^2 + (x + 3z)^2 + (x - 3z)^2 + (x - 3)^2$*.)*

Problem 2. In this problem you will derive an important result in an area of statistics. This is the theory of *lines of best fit* and plays a prominent role in data analysis, machine learning, quantitative methods in the natural sciences, among many other quantitative fields.

Background

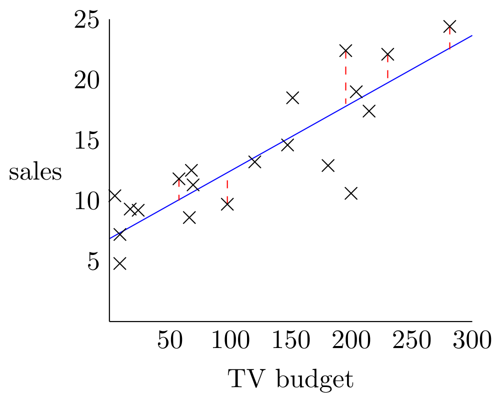
Consider a collection of n items of data of the form

$$(x_1, y_1), \ldots, (x_n, y_n) \in \mathbb{R}^2$$

We obtain a *scatterplot* by plotting this data in the xy-plane. For example, the following table represents (real) data relating twenty company's total television advertising budgets (in thousands of dollars) to sales of their product (in thousands of units)

TV Budget	Sales
230.1	22.1
44.5	10.4
17.2	9.3
151.5	18.5
180.8	12.9
8.7	7.2
57.5	11.8
120.2	13.2
8.6	4.8
199.8	10.6
66.1	8.6
214.7	17.4
23.8	9.2
97.5	9.7
204.1	19
195.4	22.4
67.8	12.5
281.4	24.4
69.2	11.3
147.3	14.6

Let x_i denote the TV budget of company i (in thousands of dollars) and y_i denote the sales of the product made by company i (in thousands of units). The resulting scatterplot is given below:



Each × represents a data point. Also represented on the scatterplot is the *(least squares) line of best fit*: this is the unique line $y = mx + b$ that *minimises* the average of the sum of the squares of the lengths of the red dashed lines. This quantity is known as the *root mean square error (RMSE)*:

$$\text{RMSE} = \sqrt{\frac{(y_1 - mx_1 - b)^2 + (y_2 - mx_2 - b)^2 + \ldots + (y_n - mx_n - b)^2}{n}} = \sqrt{\frac{1}{n}\sum_{i=1}^{n}(y_i - mx_i - b)^2}$$

To determine the (least squares) line of best fit we have to find m, b that minimises the RMSE. To do this, it is sufficient to determine the global minimum of the function

$$f(m, b) = (y_1 - mx_1 - b)^2 + (y_2 - mx_2 - b)^2 + \ldots + (y_n - mx_n - b)^2 = \sum_{i=1}^{n}(y_i - mx_i - b)^2$$

Let $\{(x_1, y_1), \ldots, (x_n, y_n)\}$ be a given data set. In particular, all the x_i's and y_i's are numerical values (i.e. constants).

a. Show that $f(m, b)$ has a unique critical point (m_0, b_0), where

$$m_0 = \frac{(\sum_{i=1}^{n} x_i y_i) - n(\overline{x})(\overline{y})}{(\sum_{i=1}^{n} x_i^2) - n\overline{x}^2}, \qquad b_0 = \overline{y} - m_0\overline{x}$$

Here

$$\overline{x} = \frac{1}{n}\sum_{i=1}^{n} x_i, \qquad \overline{y} = \frac{1}{n}\sum_{i=1}^{n} y_i$$

are the mean averages of the x_i's and y_i's.

b. Show that (m_0, b_0) is a local minimum using the second derivative test.

c. Explain why this local minimum must be a global/absolute minimum for $f(m, b)$.

d. For the TV advertising/sales data given above show that the slope of the (least squares) line of best fit is positive. Does TV advertising work? Justify your response.

Problem 3. Double integrals can often be estimated (or even exactly computed) using properties and symmetries, without resorting to evaluating iterated integrals.

a. Let D be the region inside the unit circle centered at the origin, let R be the right half of D, and let B be the bottom half of D. Decide, without calculating the value of any of the integrals, whether each integral is positive, negative, or zero. Explain your answers.

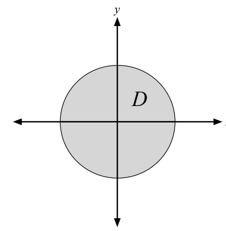
(i) $\iint_D 1\, dA$

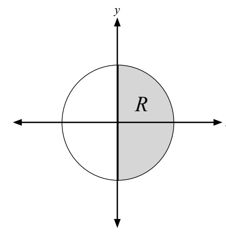
(ii) $\iint_R 5x\, dA$

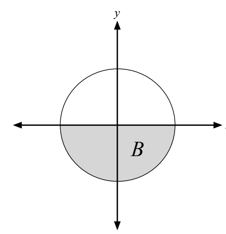
(iii) $\iint_B 5x\, dA$

(iv) $\iint_D e^x\, dA$

(v) $\iint_R x^2 y^3\, dA$







b. Compute the exact value of each double integral below *without evaluating any iterated integrals*, and explain your answers.

(i) Let $\mathcal{R}_1$ be any disk of radius 5 in the xy-plane. Compute

$$\iint_{\mathcal{R}_1} 17\,\mathrm{d}A\,.$$

(ii) Let $\mathcal{R}_2$ be a square with side length 6 and sides parallel to the x- and y-axes, centered at some point along the y-axis. Compute

$$\iint_{\mathcal{R}_2} (x+2)\,\mathrm{d}A\,.$$

(iii) Let $\mathcal{R}_3$ be any square with side length 2π and sides parallel to the x- and y-axes. Compute

$$\iint_{\mathcal{R}_3} e^x \sin(y)\,\mathrm{d}A\,.$$

(iv) Let $\mathcal{R}_4$ be a disk of radius $k > 0$ centered at the origin in the xy-plane. Compute

$$\iint_{\mathcal{R}_4} \sin^2(x) + \tan^3(xy) + \sin^2\left(x - \frac{\pi}{2}\right)\,\mathrm{d}A\,.$$

Problem 4. In this problem you will think about *changing the order of integration* and how that decision is influenced by the shape of a 2D region.

a. Consider the double integral

$$I = \int\int_R \frac{e^{x/y}}{y} dA = \int_0^1 \int_x^{\sqrt{x}} \frac{e^{x/y}}{y} dy dx\,.$$

(i) Sketch the region R.

(ii) What issue arises when you try to compute I using the indicated iterated integral?

(iii) By changing the order of integration, determine I.

b. For each of the following double integrals, sketch the region of integration R. Then evaluate the integral **by hand**, explaining your process.

(i) $\displaystyle\int_0^1 \int_0^1 \sin(e^x)\, dx\, dy + \int_1^e \int_{\ln(y)}^1 \sin(e^x)\, dx\, dy$

(ii) $\displaystyle\int_0^2 \int_{x^2}^4 x\cos\big(y^2\big)\, dy\, dx$

(iii) $\displaystyle\int_0^1 \int_2^4 \int_{-3}^1 3\, dz\, dx\, dy$

Problem 5. Since changing the order of integration corresponds to changing the order of certain limits, it should not be surprising to know that this operation holds for functions that are 'not too badly behaved'.

Fubini's Theorem: *if $f(x,y)$ is continuous on $R = [a,b] \times [c,d]$ then*

$$\int\int_R f(x,y)dA = \int_a^b \int_c^d f(x,y)dydx = \int_c^d \int_a^b f(x,y)dxdy$$

In this problem, you will see an instance where changing the order of integration does change the value of a double integral.

a. Show that

$$\frac{\partial}{\partial x}\left(\frac{-x}{(x+y)^2}\right) = \frac{x-y}{(x+y)^3}, \qquad \frac{\partial}{\partial y}\left(\frac{y}{(x+y)^2}\right) = \frac{x-y}{(x+y)^3}$$

b. By evaluating both double integrals, show that

$$\int_0^1 \int_0^1 \frac{x-y}{(x+y)^3}dxdy \neq \int_0^1 \int_0^1 \frac{x-y}{(x+y)^3}dydx$$

c. Explain why $f(x,y) = \frac{x-y}{(x+y)^3}$ is not continuous on $[0,1] \times [0,1]$.

d. (Don't Hand In): If you are curious, use Riemann Sums to show that even though we can't use Fubini's Theorem,

$$\int\int_R f(x,y)dA = 0.$$

Problem 6. The Ice-Cream Integral

Consider the solid region above the cone with equation $z = \sqrt{x^2+y^2}$, and below the unit sphere centered at the origin.

a. Sketch the solid region on a 3D coordinate axis.

b. Set up an integral in Cartesian coordinates which represents the volume of this solid.

c. Set up an iterated integral in polar coordinates that represents the volume of this solid.

d. Which of the integrals above is simpler to evaluate? Describe why, in a few sentences. For that integral, compute the integral to find the volume of this solid.