This is the sixth homework assignment for Math 160 and it is broken into two parts. The first part of the homework consists of exercises you should do (and I'll expect you to do) but you needn't turn in. As these exercises will not be graded, if you would like help with them or just want to make sure you're doing them correctly, you should (always) feel free to come to office hours (or the nightly TA sessions). The second part is the part you are expected to turn in. More precisely, please complete all problems in Part 2, write up clear and thorough solutions for them (consistent with the directions given in the syllabus[1]) and hand them in. Your write-ups are due on **Thursday, October 30th** in the box outside my office door. As always, please come and see me early if you get stuck on any part of this assignment. I am here to help!

Part 1 (Do not turn in)

Exercise 1. Please do the following:

a. For each pair of vectors below, find their cross product $\mathbf{u} \times \mathbf{v}$ and verify it is perpendicular to both $\mathbf{u}$ and $\mathbf{v}$.

(i) $\mathbf{u} = \langle 6, 0, -2 \rangle$ and $\mathbf{v} = \langle 0, 8, 0 \rangle$

(ii) $\mathbf{u} = \mathbf{i} + 3\mathbf{j} - 2\mathbf{k}$ and $\mathbf{v} = -\mathbf{i} + 5\mathbf{k}$

(iii) $\mathbf{u} = \mathbf{i} - \mathbf{j} - \mathbf{k}$ and $\mathbf{v} = \frac{1}{2}\mathbf{i} + \mathbf{j} + \frac{1}{2}\mathbf{k}$

b. The standard basis vectors have particularly nice cross products, e.g.

- $\mathbf{i} \times \mathbf{j} = \mathbf{k}$,
- $\mathbf{j} \times \mathbf{k} = \mathbf{i}$, and
- $\mathbf{k} \times \mathbf{i} = \mathbf{j}$.

Use these facts and the properties of cross products (as opposed to the cross product formula) to simplify the following expressions.

(i) $(\mathbf{i} \times \mathbf{j}) \times \mathbf{k}$

(ii) $\mathbf{k} \times (\mathbf{i} - 2\mathbf{j})$

(iii) $(\mathbf{j} - \mathbf{k}) \times (\mathbf{k} - \mathbf{i})$

c. State whether each expression is defined, and if so, whether the result is a vector or a scalar.

(i) $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})$

(ii) $\mathbf{u} \times (\mathbf{v} \cdot \mathbf{w})$

(iii) $\mathbf{u} \times (\mathbf{v} \times \mathbf{w})$

(iv) $\mathbf{u} \cdot (\mathbf{v} \cdot \mathbf{w})$

(v) $(\mathbf{u} \cdot \mathbf{v}) \times (\mathbf{u} \cdot \mathbf{w})$

(vi) $(\mathbf{u} \times \mathbf{v}) \cdot (\mathbf{u} \times \mathbf{w})$

Exercise 2. Please do the following:

a. Find the equation of the plane through the origin with normal vector $\langle 5, -1, -1 \rangle$.

b. Find the equation of the plane through the point $(5, 3, 5)$ with normal vector $\mathbf{i} + 4\mathbf{j} + \mathbf{k}$.

c. Sketch each plane using the x-, y-, and z-intercepts.

(i) $2x + 5y + z = 10$

(ii) $3x + y + 2z = 6$

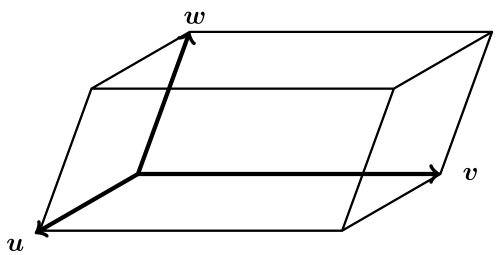
[1] Now is a superb time to read the syllabus.

(iii) $6x - 3y + 4z = 6$

Exercise 3 (Volume of slanted box)**.** Consider 3 vectors in $\mathbb{R}^3$, $\boldsymbol{u} = (1, 2, 0)$, $\boldsymbol{v} = (5, 4, -1)$, $\boldsymbol{w} = (1, 1, 1)$.

a. Find the area of the parallelogram defined by $\boldsymbol{u}$ and $\boldsymbol{v}$.

b. Similar to how $\boldsymbol{u}$ and $\boldsymbol{v}$ define a parallelogram, the vectors $\boldsymbol{u}$, $\boldsymbol{v}$, and $\boldsymbol{w}$ define a solid that looks like a slanted box. Let $\boldsymbol{u}$ and $\boldsymbol{v}$ define the base of this box. Find the height of the box.



c. Volume is given by area of the base times the height. Find the volume of the box.

d. Based on the procedure above, find a general formula for the volume of a slanted box defined by 3 distinct vectors $\boldsymbol{u}$, $\boldsymbol{v}$ and $\boldsymbol{w}$.

Part 2 (Solutions for these problems are due in the appropriate box outside my office door at 11:00AM on October 30th)

Problem 1 (Vector-valued functions, parameterizations and minimal distances)**.** Though this problem seems long, the text is mostly discussion and is dedicated to walking you through some computations (which you are asked to then repeat/generalize). So please don't be discouraged! This problem is straightforward; all you have to do is the numbered items.

A vector-valued function is a function $\mathbf{r}$ mapping from the real line $\mathbb{R}$ into $\mathbb{R}^2$ or $\mathbb{R}^3$. These functions are often given by

$$\mathbf{r}(t) = (x(t), y(t))$$

or

$$\mathbf{r}(t) = (x(t), y(t), z(t))$$

defined for real numbers t, i.e., for $t \in \mathbb{R}$. Here $x(t), y(t)$ and $z(t)$ are simply real-valued (scalar-valued) functions of t. You should think about these vector-valued functions as specifying a position in space for each time t. In fact, vector-valued functions are some of the main objects studied in classical dynamics (e.g., celestial mechanics) as they describe the motion of an object in space as a function of time. Each such vector-valued function $\mathbf{r}$ parameterizes (traces out) a curve $\mathcal{C}$ defined by

$$\mathcal{C} = \{\mathbf{r}(t) : t \in \mathbb{R}\}.$$

We shall say that $\mathbf{r}$ is a parameterization of $\mathcal{C}$. In this course, $\mathcal{C}$ will generally be a one-dimensional[2] subset of $\mathbb{R}^2$ or $\mathbb{R}^3$. Let's consider, for example, the function $\mathbf{r} : \mathbb{R} \to \mathbb{R}^2$ defined by

$$\mathbf{r}(t) = (\cos(t), \sin(t))$$

for $t \in \mathbb{R}$. To get a picture of what curve this parameterizes in the xy-plane, observe that

$$(x(t))^2 + (y(t))^2 = \cos^2(t) + \sin^2(t) = 1$$

[2]It's not always the case the these curves are one-dimensional, which is a rather curious phenomenon. If this is of interest to you, look up space-filling curves or fractal geometry.

for any t. This is to say that, for all t, $\mathbf{r}(t)$ lives on the unit circle, i.e., the curve

$$\mathcal{C} = \{\mathbf{r}(t) : t \in \mathbb{R}\}$$

parameterized by $\mathbf{r}$ is the unit circle. More is true. In fact, if you plot this curve by letting t increase and plot $\mathbf{r}(t)$ in $\mathbb{R}^2$, you'll see that the unit circle is "traced out" by moving the points at $\mathbf{r}(t)$ counterclockwise around the circle: t is the angle subtended by the red line and the x-axis (in radians). Figure 1 illustrates this.

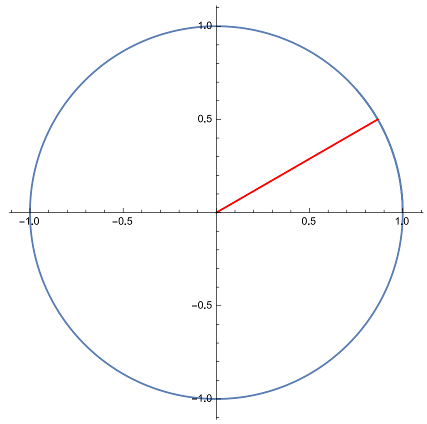


Figure 1: The curve parameterized by $\mathbf{r}(t) = (\cos(t), \sin(t))$.

Now it's your turn:

a. Plot the curve in $\mathbb{R}^3$ parameterized by

$$\mathbf{r}(t) = (\cos(t), \sin(t), t)$$

for $t \in \mathbb{R}$. Hint: If you "projected/collapsed" this curve into the xy plane, you would get a circle because the first two components are exactly what we saw in the example above. So, if you were to walk way up on the z-axis and look down, you would essentially see the unit circle. To plot the full curve, think now about what's happening in the z-direction as t increases.

We now focus on a special type of vector-valued functions, those which parameterize lines in $\mathbb{R}^2$ and $\mathbb{R}^3$. Let's first focus on the 2-dimensional case. Given constants m and b, consider

$$\mathbf{r}(t) = (t, mt + b)$$

defined for $t \in \mathbb{R}$. We note here that $x = t$ and $y = mt + b$. By making a replacement of t for x into y we obtain

$$y = mx + b$$

which is, of course, the equation of a line. There is another, perhaps more geometric, way to plot a line in $\mathbb{R}^2$. If we begin with a point (or vector) $\mathbf{r}_0 = (x_0, y_0)$ and a direction vector $\mathbf{v} = (p, q)$ we can consider

$$\mathbf{r}(t) = t\mathbf{v} + \mathbf{r}_0 = t(p, q) + (x_0, y_0) = (pt + x_0, qt + y_0)$$

defined for $t \in \mathbb{R}$. Note that, when $t = 0$, $\mathbf{r}(0) = 0 \cdot \mathbf{v} + \mathbf{r}_0 = \mathbf{r}_0$, so the curve parameterized by $\mathbf{r}$ contains the point $\mathbf{r}_0$. It is not too terribly difficult to see that this parameterizes the line through $\mathbf{r}_0$ in the direction of $\mathbf{v}$.

b. Using the construction above, find the parameterization of the line through $(0, 1)$ in the direction $(1, 1)$. Draw this line in $\mathbb{R}^2$ and indicate where the point $\mathbf{r}(0)$ and $\mathbf{r}(1)$ are on this line.

c. Given two points (a, b) and (c, d) in $\mathbb{R}^2$, find a parameterization $\mathbf{r}(t)$ of the line through through the points $P = (a, b)$ and $Q = (c, d)$. Hint: The direction vector from $P = (a, b)$ to $Q = (c, d)$ is $\mathbf{v} = (c - a, d - b)$.

Now that we can describe lines parametrically in $\mathbb{R}^2$, we can, in fact, ask some interesting questions. For example, given the line parameterized by $\mathbf{r}(t) = t(1,1) + (0,-6)$, can we find a point on the line which is closest to the point $(1,1)$? In other words, among all point on the line parameterized by $\mathbf{r}(t)$, can we find the point that minimizes the distance from this line to the point $(1,1)$?

Sure, we can! Given any t, the distance from the point $(1,1)$ to the point (on the line) $\mathbf{r}(t)$ is

$$d(t) = \|\mathbf{r}(t) - (1,1)\| = \|(t, t-6) - (1,1)\| = \|(t-1, t-7)\| = \sqrt{(t-1)^2 + (t-7)^2}.$$

We are interested in where this distance is minimized, i.e., for which t is $d(t)$ the smallest. A moment's thought shows that $d(t)$ is minimized if and only if $(d(t))^2 = d^2(t)$ is minimized, so our goal is to minimize the function

$$d^2(t) = (t-1)^2 + (t-7)^2 = 2t^2 - 16t + 50.$$

This is, of course, a Calculus 1 problem and the minimum is easily found by finding the function's critical point. We have

$$0 = \frac{d}{dt} d^2(t) = \frac{d}{dt}\left(2t^2 - 16t + 50\right) = 4t - 16$$

which yields $t = 4$ as a critical point. It is easy (with the second derivative test) to check that this is a minimum. Consequently, the function $d(t)$ is minimized when $t = 4$ from which we obtain the minimum distance

$$d(4) = \sqrt{(4-1)^2 + (4-7)^2} = \sqrt{9+9} = 3\sqrt{2}.$$

which is attained at the point

$$\mathbf{r}(4) = (4, 4-6) = (4, -2).$$

Now, it's your turn to extend these ideas into $\mathbb{R}^3$.

d. Given a point $\mathbf{r}_0 = (x_0, y_0, z_0)$ and a direction $\mathbf{v} = (p, q, r)$ find a parameterization of the line through $\mathbf{r}_0$ in the direction $\mathbf{v}$.

e. Given points (a, b, c) and (d, e, f), find a parameterization of the line through (a, b, c) and (d, e, f).

f. Find the minimum distance from the line through $(-1, 0, 0)$ and $(0, 1, 2)$ to the point $(0, 0, 7)$. Find the point at which this minimum distance is attained.

Problem 2. In this problem you will determine a function $D(u, v, w)$ that computes the distance from an arbitrary point $(u, v, w) \in \mathbb{R}^3$ to an arbitrary plane $\mathcal{P}$. Here, *distance* means *shortest distance*.

a. Determine the distance from $P = (u, v, w)$ to each of the coordinate planes (i.e. the xy-, xz- and yz-planes). Your answer should be in terms of u, v, w.

b. Consider the plane defined by

$$2x - 4y + 3z = 2.$$

Determine the distance from $P = (u, v, w)$ to this plane. Your answer should be in terms of u, v, w.

c. Consider a general plane

$$\mathcal{P}: \quad \mathbf{n} \cdot (x, y, z) = d$$

Here $\mathbf{n} = (a, b, c)$ is a normal vector for the plane. Determine a formula for the function

$$D : \mathbb{R}^3 \to \mathbb{R}\ ,\ (u, v, w) \mapsto D(u, v, w)$$

where $D(u, v, w) =$ distance from (u, v, w) to $\mathcal{P}$. Your answer should be in terms of $\mathbf{n}$ and $\mathbf{u} = (u, v, w)$.

Problem 3 (What defines a plane?)**.** For each condition below, determine if it is satisfied by

- no planes in $\mathbb{R}^3$,

- a unique plane in $\mathbb{R}^3$, or
- infinitely-many planes in $\mathbb{R}^3$.

For each condition that determines a unique plane, find an equation for that plane.

a. The plane P contains the points $(2, 0, 1)$, $(3, 2, 1)$, and $(-1, -1, -1)$.

b. The plane P contains the origin and is perpendicular to $\langle 2, 0, -3\rangle$ and $\langle -4, 0, 6\rangle$.

c. The plane P is parallel to vectors $\langle 2, 0, 2\rangle$ and $\langle 1, 1, 1\rangle$.

d. The plane P contains the origin and is perpendicular to $\langle 1, 1, 1\rangle$ and $\langle -1, 3, 2\rangle$.

e. The plane P contains the point $(3, -1, 0)$ and has a normal vector parallel to the z-axis.

f. The plane P contains the points $(-1, -2, 1)$, $(3, 6, -3)$, and $(-2, -4, 2)$.

g. The plane P contains the point $(2, 0, 3)$ and is parallel to the plane with equation $x + 2y + z = 1$.

Problem 4. a. Two planes in $\mathbb{R}^3$ may or may not intersect. How can you distinguish one case from another using the normal vectors of the two planes?

b. If two distinct planes in $\mathbb{R}^3$ intersect, the intersection in a line. Consider the planes given by the equations

$$3x - y + 2z = 3$$

$$x + 4z = 2.$$

Use the cross product to find a direction vector $\mathbf{v}$ for the line of intersection (see Problem 2 for the definition of the direction vector of a line). Explain why your computation works.

c. Consider vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ in $\mathbb{R}^3$. If we know that

$$(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} = \mathbf{0},$$

what do we know about the geometry of $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$?

d. Suppose $\mathcal{P}$, $\mathcal{Q}$, and $\mathcal{R}$ are three different planes in $\mathbb{R}^3$ with normal vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ respectively. There are a number possibilities for the intersection $\mathcal{P} \cap \mathcal{Q} \cap \mathcal{R}$ of all three (that is, the set of points (x, y, z) which are in all three planes). For each of the possibilities listed below, explain the relationship between the vectors $\mathbf{u}, \mathbf{v}$,$\mathbf{w}$, and $\mathbf{u} \times \mathbf{v}$, as well as the scalar $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}$ (using terms like "zero" or "not zero" or "parallel" or "lie in the same plane"). Sketch each situation.

i. $\mathcal{P} \cap \mathcal{Q}$, $\mathcal{P} \cap \mathcal{R}$, $\mathcal{Q} \cap \mathcal{R}$ are all lines and $\mathcal{P} \cap \mathcal{Q} \cap \mathcal{R}$ is a single point.

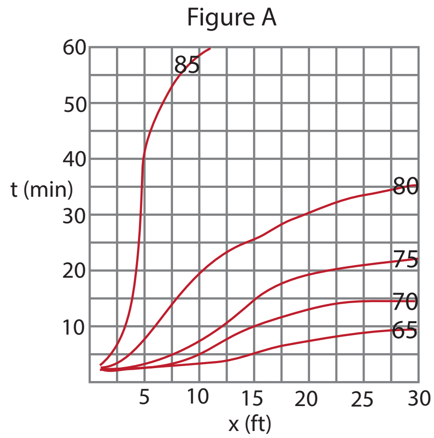
ii. $\mathcal{P} \cap \mathcal{Q}$, $\mathcal{P} \cap \mathcal{R}$, $\mathcal{Q} \cap \mathcal{R}$ are all lines and $\mathcal{P} \cap \mathcal{Q} \cap \mathcal{R}$ contains no points.

iii. $\mathcal{P} \cap \mathcal{Q}$, $\mathcal{P} \cap \mathcal{R}$, $\mathcal{Q} \cap \mathcal{R}$ are all lines and $\mathcal{P} \cap \mathcal{Q} \cap \mathcal{R}$ is a line.

iv. $\mathcal{P} \cap \mathcal{Q}$ contains no points, $\mathcal{P} \cap \mathcal{R}$ and $\mathcal{Q} \cap \mathcal{R}$ are lines. In this case, what is true about $\mathcal{P} \cap \mathcal{Q} \cap \mathcal{R}$?

Problem 5. In this problem, you will investigate a specific contour diagram from an unspecified function.

You are in a room 30 feet long with a heater on the wall at one end. In the morning, the temperature in the room is 65°F. You turn on the heater, which quickly warms up to 85°F. Let $H(x, t)$ be the temperature x feet from the heater t minutes after the heater is turned on. Figure A shows the contour diagram for H.



a. Using the contour diagram, about how warm is it 15 feet from the heater 20 minutes after it was turned on?

b. On a set of coordinate axes, draw and label sketches of the cross-sections $H(x, 10)$ and $H(x, 30)$. Be sure to label your axes! Describe what these two cross-sections represent. How can you explain the difference between these two graphs?

c. On another set of coordinate axes, draw and label sketches of the cross-sections $H(10, t)$ and $H(30, t)$. Be sure to label your axes! Describe what these two cross-sections represent. How can you explain the difference between these two graphs?

d. Imagine that 10 minutes after the heater is turned on, you and your cat Fermat are sitting 10 feet from the heater. How will H change as time goes on, if you stay in the same place? How do you know?

e. Imagine that 20 minutes after the heater has been turned on, Fermat gets up and walks quickly away from the heater. How will H change for Fermat as he walks? How do you know?

f. Sketch a new contour diagram for the function $T_{10}(a, b)$, which gives the temperature at point (a, b) in the room 10 minutes after the heater was turned on. Indicate the location of the heater in your picture. How does this new contour diagram differ from the original?[3]

[3] The coordinates (a, b) here are meant to be coordinates in the x-y plane - you may ignore the fact that the room has height and the fact that heat rises, and assume temporarily that heat is determined entirely by your position in the room.