This is the fifth homework assignment for Math 160 and it is broken into two parts. The first part of the homework consists of exercises you should do (and I'll expect you to do) but you needn't turn in. As these exercises will not be graded, if you would like help with them or just want to make sure you're doing them correctly, you should (always) feel free to come to office hours (or the nightly TA sessions). The second part is the part you are expected to turn in. More precisely, please complete all problems in Part 2, write up clear and thorough solutions for them (consistent with the directions given in the syllabus[1]) and hand them in. Your write-ups are due on **Thursday, October 23rd** in the box outside my office door. As always, please come and see me early if you get stuck on any part of this assignment. I am here to help!

Part 1 (Do not turn in)

Exercise 1. Please do the following:

a. What are the projections of the point (2,3,5) on the $xy-$, $yz-$, and $xz-$ planes? Draw a rectangular box with the origin and (2,3,5) as opposite vertices and with its faces parallel to the coordinate planes. Label all vertices of the box. Find the length of the diagonal of the box.

b. Determine whether the points lie on a straight line:

   (i) $A = (2, 4, 2)$, B=(3,7,-2), $C = (1, 3, 3)$

   (ii) D=(0,-5,5), $E = (1, -2, 4)$, $F = (3, 4, 2)$.

c. Find an equation of the sphere with center $(-3, 2, 5)$ and radius r. What is the intersection of this sphere with the yz-plane?

d. Show that the equation represents a sphere and find it center and radius:

   (i) $x^2 + y^2 + z^2 - 2x - 4y + 8z = 15$

   (ii) $2x^2 + 2y^2 + 2z^2 = 8x - 24z + 1$

e. Describe, in words, the following regions in $\mathbb{R}^3$:

   (i) $0 \leq z \leq 6$

   (ii) $y < 8$

   (iii) $x^2 + y^2 + z^z \leq 3$.

Exercise 2. Please do the following:

a. For each pair of points P and Q below, find the vector from P to Q and sketch the vector in $\mathbb{R}^2$ or $\mathbb{R}^3$.

   (i) $P = (-1, 1)$ and $Q = (3, 2)$

   (ii) $P = (-1, 3)$ and $Q = (2, 2)$

   (iii) $P = (0, 3, 1)$ and $Q = (2, 3, -1)$.

b. For each pair of vectors $\mathbf{v}$ and $\mathbf{w}$ below, calculate the sum $\mathbf{v} + \mathbf{w}$ and sketch all three vectors in $\mathbb{R}^2$ or $\mathbb{R}^3$.

   (i) $\mathbf{v} = \langle -1, 4\rangle$ and $\mathbf{w} = \langle 6, -2\rangle$

   (ii) $\mathbf{v} = \langle 3, 0, 1\rangle$ and $\mathbf{w} = \langle 0, 8, 0\rangle$.

c. Let $\mathbf{v} = \mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$ and $\mathbf{w} = -2\mathbf{i} - \mathbf{j} + 5\mathbf{k}$.

   (i) Find $\mathbf{v} + \mathbf{w}$.

   (ii) Find $2\mathbf{v} + 3\mathbf{w}$.

[1] Now is a superb time to read the syllabus.

(iii) Find $\|\mathbf{v}\|$.

(iv) Find $\|\mathbf{v} - \mathbf{w}\|$.

d. Find a unit vector that has the same direction as the given vector.

(i) $-3\mathbf{i} + 7\mathbf{j}$

(ii) $\langle 8, -1, 4\rangle$.

e. Let $\mathbf{v}$ be a vector with length 4 that makes an angle of $\pi/3$ with the positive x-axis. If $\mathbf{v}$ lies in the first quadrant of $\mathbb{R}^2$, find the components of $\mathbf{v}$.

Exercise 3. Please do the following:

a. Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ be vectors in $\mathbb{R}^n$. Which of the following expressions are well-defined?

(i) $(\mathbf{u} \cdot \mathbf{v}) \cdot \mathbf{w}$

(ii) $(\mathbf{u} \cdot \mathbf{v})\mathbf{w}$

(iii) $\|\mathbf{v}\|(\mathbf{u} \cdot \mathbf{w})$

(iv) $\|\mathbf{v}\| \cdot (\mathbf{u} \cdot \mathbf{w})$

(v) $\mathbf{v} \cdot (\mathbf{u} + \mathbf{w})$

(vi) $\mathbf{u} \cdot \mathbf{v} + \mathbf{w}$

b. For each pair of vectors $\mathbf{v}$ and $\mathbf{w}$ below, calculate their dot product $\mathbf{v} \cdot \mathbf{w}$.

(i) $\mathbf{v} = \langle -2, \frac{1}{3}\rangle$ and $\mathbf{w} = \langle -5, 12\rangle$

(ii) $\mathbf{v} = 2\mathbf{i} + \mathbf{j}$ and $\mathbf{w} = \mathbf{i} - \mathbf{j} + \mathbf{k}$

(iii) $\|\mathbf{v}\| = 6$, $\|\mathbf{w}\| = 5$, and the angle between $\mathbf{v}$ and $\mathbf{w}$ is $2\pi/3$.

c. For each pair of vectors $\mathbf{v}$ and $\mathbf{w}$ below, calculate the projection of $\mathbf{w}$ onto $\mathbf{v}$.

(i) $\mathbf{v} = \langle -5, 12\rangle$ and $\mathbf{w} = \langle 4, 6\rangle$

(ii) $\mathbf{v} = \langle 3, 6, -2\rangle$ and $\mathbf{w} = \langle 1, 2, 3\rangle$

(iii) $\mathbf{v} = 2\mathbf{i} - \mathbf{j} + 4\mathbf{k}$ and $\mathbf{w} = \mathbf{j} + \frac{1}{2}\mathbf{k}$.

Part 2 (Solutions for these problems are due in the appropriate box outside my office door at 11:00AM on October 23rd)

In the first problem, we will make use of the following theorem (essentially, Theorems 8+9 from Stewart's text):

Theorem A. *Let f be infinitely differentiable on a neiborhood of the form $(a-R, a+R)$ where a is a fixed "center" and $R > 0$. Suppose that, for some constant $M > 0$,*

$$\left|f^{(n)}(x)\right| \leq M$$

for every x in the interval $(a-R, a+R)$ (i.e., for all x for which $|x-a| < R$) and all integers n. Then f is equal to its Taylor series centered at a, i.e.,

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

and this holds for all x such that $|x-a| < R$.

The moral of this theorem is the following: If you can uniformly bound all of the derivatives of a function on an interval centered at some a, then the function is equal to its Taylor series centered at a on the whole of the interval. In particular, the Taylor series must converge on that interval.

Problem 1 (Some Applications of Taylor Series)**.** In this problem, we use results of Taylor series to obtain two important identities for common functions.

a. Consider $f(x) = e^x$ and, for a fixed a, its Taylor expansion

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n;$$

note that, when a is non-zero, this is different from the definition/Maclaurin expansion for e^x. Please do the following:

(i) For $R > 0$, show that there is some M for which $\left|f^{(n)}(x)\right| \leq M$ for all $x \in (a-R, a+R)$ and $n = 1, 2, \ldots$. What is M in terms of a and R?

(ii) Using the theorem, conclude that $f(x)$ is equal to its Taylor series on the interval $(a-R, a+R)$. In fact, argue that since this actually held for every R,

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

for all real numbers x.

(iii) Compute this Taylor series and simplify as much as possible.

(iv) Conclude that, for any real numbers a and b,

$$e^{a+b} = e^a e^b.$$

b. Consider $g(x) = \cos(x)$ and, for a fixed a, its Taylor expansion

$$\sum_{n=0}^{\infty} \frac{g^{(n)}(a)}{n!}(x-a)^n.$$

(i) For $R = \infty$, show that there is some M for which $\left|g^{(n)}(x)\right| \leq M$ for all $x \in (a-R, a+R) = \mathbb{R}$ and $n = 1, 2, \ldots$. What is M?

(ii) Compute all derivatives of g at our fixed a. Your answer should be in terms of sines and cosines evaluated at our arbitrary (but fixed) point a.

(iii) Since the conditions of the theorem are satisfied for g, we have

$$g(x) = \sum_{n=0}^{\infty} \frac{g^{(n)}(a)}{n!}(x-a)^n$$

for all $x \in (a-R, a+R) = \mathbb{R}$. Plug in all the derivatives you computes for g, by grouping sines and cosines, express your series in the form

$$g(x) = \cos(a)\sum_{n=0}^{\infty} a_n(x-a)^n + \sin(a)\sum_{n=0}^{\infty} b_n(x-a)^n.$$

Do you recognize the two series left over? Identify them.

(iv) Plug in $x = a + b$ to obtain a common trigonometric identity.

Problem 2 (Working in $\mathbb{R}^3$)**.** a. In $\mathbb{R}^2$, the Pythagorean theorem gives the distance between two points (a_1, a_2) and (b_1, b_2) as

$$d = \sqrt{(a_1-b_1)^2 + (a_2-b_2)^2}.$$

Using the Pythagorean theorem twice, show that the distance between two points (a_1, a_2, a_3) and (b_1, b_2, b_3) in $\mathbb{R}^3$ is given by

$$d = \sqrt{(a_1 - b_1)^2 + (a_2 - b_2)^2 + (a_3 - b_3)^2}.$$

It may be helpful to draw a cube and think of this distance as the distance between the two farthest corners of the cube.

b. Given a fixed positive number d and a fixed point (a, b, c) in $\mathbb{R}^3$, describe the set

$$\left\{(x, y, z) \in \mathbb{R}^3 : \sqrt{(x-a)^2 + (y-b)^2 + (z-c)^2} = d\right\}.$$

Sketch, in $\mathbb{R}^3$, this set in the special case that $d = 1$ and $(a, b, c) = (1, 1, 1)$. In this, you should note that you're actually drawing the set/locus of points (x, y, z) that satisfies the 3-variable equation

$$(x-1)^2 + (y-1)^2 + (z-1)^2 = 1.$$

c. Equations of 2 variables representing curves in $\mathbb{R}^2$ become equations of surfaces in $\mathbb{R}^3$.

(i) In 2-space, $x^2 + y^2 = 4$ is a circle of radius 2. What shape does this give in 3-space? Sketch the shape.

(ii) What shape is $x^2 + z^2 = 4$ in 3-space? Sketch the shape.

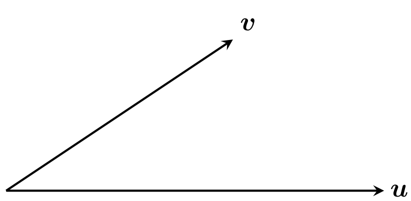
(iii) In 2-space, $y = x^2$ is a parabola. Sketch and describe the shape in 3-space.

For the remainder of the problem set, we will denote vectors (in $\mathbb{R}^2$ and $\mathbb{R}^3$) in bold letters, e.g., $\mathbf{v}, \mathbf{w}, \mathbf{a}, \mathbf{a}$ as opposed to having arrows as you might have seen in lecture (e.g., $\vec{u}$). Also, recall that the magnitude (or norm) of a vector $\mathbf{v}$ is denoted by $\|\mathbf{v}\|$. For example, for a vector $\mathbf{v} = \langle v_1, v_2, v_3\rangle$ in $\mathbb{R}^3$,

$$\|\mathbf{v}\| = \sqrt{v_1^2 + v_2^2 + v_3^2}.$$

Problem 3 (Vectors)**.** In this problem, we study some basic operations and geometric notions for vectors.

a. Consider two nonzero vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ oriented in the following way.



Make sketches to represent the following quantities:

(i) $2\boldsymbol{u}$

(ii) $-\boldsymbol{v}$

(iii) $\boldsymbol{u} + \boldsymbol{v}$

(iv) $\boldsymbol{u} - \boldsymbol{v}$

(v) $2\boldsymbol{v} - \boldsymbol{u}$

b. Letting $\boldsymbol{u} = \langle 3, -1, 2\rangle$ and $\boldsymbol{v} = \langle 1, 0, 4\rangle$, compute the quantities from part (a).

c. Let $\boldsymbol{a} = \langle 2, 0, -1\rangle$ and $\boldsymbol{b} = \langle -3, 4, 1\rangle$.

(i) Find $\|\boldsymbol{a}\|$.

(ii) Find a unit vector in the direction of $\boldsymbol{b}$.

(iii) Find a vector parallel to $\boldsymbol{b}$ that has length 2.

(iv) Find $\cos(\theta)$ where θ is the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$.

(v) Find a vector perpendicular to $\boldsymbol{a}$ that has length 3.

(vi) Find all possible values of x and y so that $\langle x, y, 3\rangle$ is perpendicular to $\boldsymbol{b}$.

Given vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^3$ with $\mathbf{v}$ non-zero, we define the *vector projection of* $\mathbf{u}$ *onto* $\mathbf{v}$ to be the vector

$$\operatorname{Proj}_{\mathbf{v}} \mathbf{u} = \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2}\right) \mathbf{v}.$$

Geometrically, $\operatorname{Proj}_{\mathbf{v}} \mathbf{u} = \left(\|\mathbf{u}\| \cos\theta) \frac{\mathbf{v}}{\|\mathbf{v}\|}\right)$, where θ is the angle between $\mathbf{u}$ and $\mathbf{v}$:

Problem 4 (The Vector Projection (and some of its properties))**.** In this problem, we explore the vector projection and its properties.

a) For the following vectors, compute $\operatorname{Proj}_{\mathbf{v}} \mathbf{u}$. Also, draw the vectors and explain why the result makes sense.

  i) $\mathbf{v} = (1, 0, 0)$ and $\mathbf{u} = (2, 3, 4)$

  ii) $\mathbf{v} = (1, 1, 0)$ and $\mathbf{u} = (1, 1, 1)$.

  iii) $\mathbf{v} = (1, 1, 1)$ and $\mathbf{u} = (1, 1, 0)$.

  iv) $\mathbf{v} = (1, -1)$ and $\mathbf{u} = (1, 1)$.

b) Let's now consider general vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^3$ with $\mathbf{v}$ non-zero. Define $\mathbf{w} = \operatorname{Proj}_{\mathbf{v}} \mathbf{u}$. Using properties of the dot product, show that the $\mathbf{w}$ is independent of the length of $\mathbf{v}$ so as long as $\mathbf{v}$ is non-zero. Specifically, show that, for any $c \neq 0$, the projection of $\mathbf{u}$ on $c\mathbf{v}$ is equal to $\mathbf{w}$, i.e., show

$$\operatorname{Proj}_{c\mathbf{v}} \mathbf{u} = \operatorname{Proj}_{\mathbf{v}} \mathbf{u} = \mathbf{w}.$$

c) Show that $\mathbf{v}$ and $\mathbf{z} = \mathbf{u} - \mathbf{w}$ are orthogonal by computing their dot product.

d) Sketch the triangle with sides $\mathbf{u}$, $\mathbf{w}$ and $\mathbf{z}$. Is this a right triangle?

e) Taking this a little further (and using the ideas of vector addition – tip-to-tail), argue that $\mathbf{u}$ can be written as a sum of two vectors, one which is parallel to $\mathbf{v}$ and one which is perpendicular/orthogonal to $\mathbf{v}$.

f) Going back to Item a), write $\mathbf{u}$ as a sum of vectors, one which is parallel to $\mathbf{v}$ and one which is perpendicular to $\mathbf{v}$ for i) and ii) of a).