This is the first homework assignment for Math 160 and it is broken into two parts. The first part of the homework consists of exercises you should do (and I'll expect you to do) but you needn't turn in. As these exercises will not be graded, if you would like help with them or just want to make sure you're doing them correctly, you should (always) feel free to come to office hours (or the nightly TA sessions). The second part is the part you are expected to turn in. More precisely, please complete all problems in Part 2, write up clear and thorough solutions for them (consistent with the directions given in the syllabus[1]) and hand them in. Your write-ups are due on **Thursday, September 18th** in the box outside my office door. As always, please come and see me early if you get stuck on any part of this assignment. I am here to help!

Part 1 (Do not turn in)

Exercise 1 (This week's reading)**.** Please do the following:

a. Read Chapter 3.1-3.4, 3.6-3.7 of A Short Book on Long Sums by Fernando Gouvêa.

b. Read Sections 11.2-11.4 in Multivariable Calculus by James Stewart.

Exercise 2. Please do the following sub-exercises:

a. Answer the following:

  (i) What is the difference between a sequence and a series?

  (ii) What does it mean for a series to converge? What does it mean for a series to diverge?

  (iii) Explain what it means to say that

$$\sum_{n=1}^{\infty} a_n = 5\,?$$

b. Consider the series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n!}$.

  (i) Calculate the first six partial sums of the series.

  (ii) Does the series appear to converge or diverge?

c. Let $b_n = \dfrac{2n}{3n+1}$.

  (i) Does the sequence b_n converge?

  (ii) Does the series $\displaystyle\sum_{n=1}^{\infty} b_n$ converge?

Exercise 3. Determine whether each geometric series below converges or diverges. If it converges, find its sum.

a. $3 - 4 + \frac{16}{3} - \frac{64}{9} + \cdots$

b. $4 + 3 + \frac{9}{4} + \frac{27}{16} + \cdots$

c. $10 - 2 + 0.4 - 0.08 + \cdots$

d. $2 + 0.5 + 0.125 + 0.03125 + \cdots$

e. $\displaystyle\sum_{n=1}^{\infty} 6(0.9)^{n-1}$

[1] Now is a superb time to read the syllabus.

f. $\sum_{n=1}^{\infty} \frac{10^n}{(-9)^{n-1}}$

g. $\sum_{n=1}^{\infty} \frac{(-3)^{n-1}}{4^n}$

h. $\sum_{n=0}^{\infty} \frac{1}{(\sqrt{2})^n}$

Exercise 4. Determine whether each series below converges or diverges. If it converges, find its sum.

a. $\frac{1}{3} + \frac{1}{6} + \frac{1}{9} + \frac{1}{12} + \frac{1}{15} + \cdots$

b. $\frac{1}{3} + \frac{2}{9} + \frac{1}{27} + \frac{2}{81} + \frac{1}{243} + \frac{2}{729} + \cdots$

c. $\sum_{n=1}^{\infty} \frac{n-1}{3n-1}$

d. $\sum_{n=1}^{\infty} \frac{n(n+2)}{(n+3)^2}$

e. $\sum_{n=1}^{\infty} \frac{1+2^n}{3^n}$

f. $\sum_{n=1}^{\infty} \frac{1+3^n}{4^n}$

Exercise 5.

a. Consider the series $\sum_{n=1}^{\infty} (-5)^n x^n$.

(i) Find the values of x for which the series converges.

(ii) Find a formula for the sum of the series for those values of x.

b. Consider the series $\sum_{n=1}^{\infty} (x+2)^n$.

(i) Find the values of x for which the series converges.

(ii) Find a formula for the sum of the series for those values of x.

c. Consider the series $\sum_{n=1}^{\infty} \frac{2^n}{x^n}$.

(i) Find the values of x for which the series converges.

(ii) Find a formula for the sum of the series for those values of x.

Part 2 (Solutions for these problems are due in the appropriate box outside my office door at 11:00AM on September 18th)

Problem 1. In this problem, we will make use of the $\varepsilon - N$ definition of convergence of a sequence:

Definition A. *Let $\{a_n\}$ be a sequence of real numbers. We say that $\{a_n\}$ converges to a real number L if for any $\varepsilon > 0$, there is some natural number N_ε so that*

$$\text{for all } n \geq N_\varepsilon, \text{ we have } |a_n - L| < \varepsilon.$$

In this case, we say L is the limit of the sequence, and write $\lim_{n\to\infty} a_n = L$. If no such number L exists, then we say the sequence diverges.

More informally, this means that the sequence converges to L if, no matter how close we need to be to L (ε measures this closeness), there is some place in the sequence (N_ε gives us this place) beyond which we are always and forever at least that close to L. In this exercise, you will be given some values for ε, our closeness value, and you will need to find a place in the sequence (N_ε) which guarantees that closeness.

Consider the sequence

$$\{a_n\} = \left\{\frac{1}{2^n}\right\}$$

.

a. To what limit L does this sequence converge?

b. Find N so that if $n \geq N$, then a_n is within $\frac{1}{8}$ of L. That is, find $N_{\frac{1}{8}}$ Be sure to explain not only why a_N is within $\frac{1}{8}$ of L, but how you know that all the terms appearing later in the sequence will *also* be within $\frac{1}{8}$ of L.

c. Find $N_{\frac{1}{100}}$ (and again, explain).

d. Give the best general formula you can find for N_ε, and explain your reasoning.

Problem 2 (Golden Ratio)**.** In this problem we will investigate the ratios between consecutive Fibonacci numbers.

a. Let f be a continuous function and let a be some real number. Consider the sequence $\{a_n\}$ defined by the initial condition $a_1 = a$ and the recurrence relation $a_n = f(a_{n-1})$, i.e.

$$a_n = \underbrace{f(f(\cdots f(f(a))\cdots))}_{n-1 \text{ times}}.$$

Assuming that a_n converges to some limit L, prove that L is a *fixed point* of f, i.e. $f(L) = L$.

b. The sequence of *Fibonacci numbers* can be defined recursively by the initial conditions $F_1 = 0$ and $F_2 = 1$ and the recurrence relation $F_n = F_{n-2} + F_{n-1}$. If $r_n = \dfrac{F_{n+1}}{F_n}$ is the sequence of ratios of successive Fibonacci numbers, prove that $\{r_n\}$ satisfies the recurrence

$$r_n = 1 + \frac{1}{r_{n-1}}.$$

c. Assuming that the sequence $\{r_n\}$ defined above is convergent, find its limit L.

Problem 3 (Application of sequences and convergence)**.** Many numerical methods for solving mathematics problems rely on generating a sequence of numbers (iterates) which are “guesses” of the solution until a “good enough” guess is output. Consider Newton's method for root finding (described on homework 0), where the goal is to approximate a root x_* of a function $f(x)$ by the sequence $x_0, x_1, ..., x_k, \ldots$ generated by,

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}.$$

If we know x_*, one way to measure convergence of Newton's method is to verify that x_k approaches x_*, meaning $|x_k - x_*| \to 0$ as $k \to \infty$. However, we often do not know the exact value of x_* (why use Newton's method if we can directly solve for x_*?) The goal of this problem is to explore what convergence can mean for numerical methods, particularly when it is possible that x_* is not known.

a. Using the Colab link below, run the first block of code with $x_0 = 6$ to determine a root of $f(x) = x^5 - 2x^4 + x - 2$. This will display 20 iterations of Newton's method with sequences $\{x_k\}$, $\{f(x_k)\}$ and $\{|x_k - x_{k-1}|\}$. Report whether the sequences printed in the table converge or diverge.

b. One way to use Newton's method is to solve equations that can be reformulated as root problems, for example, $x = \cos(x)$. This is equivalent to finding the roots of $f(x) = x - \cos(x)$. Using the Colab link below, run the second block of code with initial point $x_0 = -1$. This will display 20 iterations of Newton's method. What is the root of the function? Report whether the sequences printed in the table converge or diverge.

c. In the previous two examples, you should see that $\{x_k\}$ converges nicely to the root within 20 iterations. However, for a general function f, the method might not always converge. Let's consider a more complicated example, where the goal is to explore how to determine when Newton's method has successfully found the root of a function, i.e., what a good condition is for convergence. Consider the function $f(x) = e^x - xe^x$.

   (i) Plot (sketch or use plotting software of your choice) and describe the features of f. Clearly label all roots, critical points, and asymptotes.

   (ii) Using f and f', write the Newton iteration formula (see Homework 0) for approximating a root of f with starting point x_0

   (iii) What do the sequences $\{x_k\}$, $\{f(x_k)\}$ and $\{|x_k - x_{k-1}|\}$ signify in the context of Newton's method? If Newton's method is successful in finding the root, do you expect these sequences to converge as $k \to \infty$? If so, what should they converge to?

   (iv) Using the Colab link below, run the third block of code. You will not need to change anything except the initial point, x_0, in the following parts. After clicking the "Run" button in the top left, the code will output 20 iterations of Newton's method, a text line saying whether Newton's method is successful or not.

   $$\text{(i)}x_0 = -5 \qquad \text{(ii)}x_0 = -1 \qquad \text{(iii)}x_0 = 10 \qquad \text{(iv)}x_0 = 2 \qquad \text{(v)}x_0 = 5.$$

   (v) As a convergence criterion, the code for this part uses x_k "close enough" to x_*, $|x_k - x_*| < 10^{-12}$, to determine whether Newton is successful or not. However, in practice, x_* is not known. From the results in parts (a) and (b), since we are looking for $f = 0$, you may think that $|f(x_k)|$ very close to zero is a good stopping criterion. However, this is not the case. Use your findings from the previous part and the geometry of the function described in part (i) to explain why. If $f(x_k) \to 0$, does this necessary mean a root is found?

   (vi) Based on your exploration, if the Newton method succeeds in finding a root, what is a better convergence criterion and why?

Colab link: https://colab.research.google.com/drive/1vXFGkVa-XgAu59Q1N5DzOkJ1IwFzllPs?usp=sharing

Problem 4. In this problem, we investigate the convergence of series via its definition, i.e., in terms of the convergence/divergence of its sequence of partial sums.

a) First, consider a series $\sum_{k=1}^{\infty} a_k$ whose partial sums $\{S_n\}_{n=1}^{\infty}$ are known to satisfy

$$0 \leq S_n \leq 100 \tag{1}$$

for $n = 1, 2, \ldots$.

i) Give an example of a sequence $\{a_k\}_{k=1}^{\infty}$ whose partial sums satisfy the above inequality (1) but for which the series $\sum_{k=1}^{\infty} a_k$ is divergent.

ii) Suppose that, in addition to the inequality (1), it is known that $a_k \geq 0$ for all $k = 1, 2, 3, \ldots$. Show that the series $\sum_{k=1}^{\infty} a_k$ must converge.

b) We now turn to a specific example. Suppose that the partial sums of a series $\sum_{k=1}^{\infty} b_k$ are given by

$$S_n = \ln\left(\frac{2n}{n+1}\right)$$

for $n = 1, 2, \ldots$.

i) Evaluate $\lim_{n\to\infty} S_n$.

ii) Does the series $\sum_{k=1}^{\infty} b_k$ converge? Explain your reasoning (no more than one sentence needed).

iii) Find a formula for the terms b_k of the series $\sum_{k=1}^{\infty} b_k$ and simplify as much as possible.

iv) Using your formula in (iii), show directly that $\lim_{k\to\infty} b_k = 0$.

v) Explain how you could have known $\lim_{k\to\infty} b_k = 0$ without using the formula you found in (iii)

c) Sometimes we can compute the partial sums exactly. Consider the infinite series

$$\sum_{k=4}^{\infty} \frac{1}{\sqrt{k}} - \frac{1}{\sqrt{k+1}}.$$

(i) By simplifying, find the partial sum

$$S_n = \sum_{k=4}^{n} \frac{1}{\sqrt{k}} - \frac{1}{\sqrt{k+1}}.$$

(ii) Does the infinite series we are considering diverge or converge? If it converges, find the sum of the series.

Problem 5. In this problem (essentially Problem 3.6.2 in Gouvêa), you will give an alternative explanation for why the harmonic series diverges, and also understand something about how quickly its partial sums grow.

Let

$$S_n = \sum_{k=1}^{n} \frac{1}{k}$$

be the nth partial sum of the harmonic series. Also, let

$$I_n = \int_1^{n+1} \frac{1}{x}\, dx$$

be the integral of $\frac{1}{x}$ from 1 to $n+1$.

a) Show that S_n is actually the left-endpoint Riemann sum for $f(x) = 1/x$ using n equal-sized subdivisions on the interval $[1, n+1]$;

b) Show that $S_n > I_n$ (a picture may be helpful), thus giving a lower bound on S_n.

c) Check that $\lim_{n\to\infty} I_n$ is infinity, and use this together with the inequality you found in previous part to show that the the harmonic series diverges.

d) Can you implement a similar idea with right Riemann sums to to find an *upper* bound for the nth partial sum S_n?

e) Explain why $\ln(n)$ is a pretty good estimate for S_n.

Problem 6. In this problem you will investigate the relationship between repeating decimals and fractions.

a) Let $x = 0.333\ldots$.

   i) Express x as a geometric series.

   ii) Determine the limit of this geometric series.

b) Let $y = 0.ababab\ldots$, where $0 \le a, b \le 9$ are integers.

   i) Express y as a geometric series.

   ii) Show that $y = \frac{10a+b}{99}$.

   iii) Use the previous problem to determine the decimal expansion of $\frac{8}{33}$.