SCALAR CURL

Suppose that $\vec{F}$ is a continuous vector field on an open region $\mathcal{U} \subset \mathbb{R}^2$. For $\vec{a} \in \mathcal{U}$, we define

$$\operatorname{scurl} \vec{F}(\vec{a}) = \lim_{C \to \vec{a}} \frac{1}{\text{Area}(C)} \int_C \vec{F} \cdot d\vec{s}$$

<u>Theorem</u> Suppose $\vec{F}(\vec{x}) = \begin{pmatrix} M(\vec{x}) \\ N(\vec{x}) \end{pmatrix}$ is C^1.

Then $\operatorname{scurl} \vec{F}(\vec{x}) = \frac{\partial}{\partial x} N(\vec{a}) - \frac{\partial}{\partial y} M(\vec{a})$

<u>proof</u> We will use two theorems from Calc I

<u>MVT for Integrals</u>: If g is continuous on $[a,b]$ then there exists $t^* \in [a,b]$ s.t.

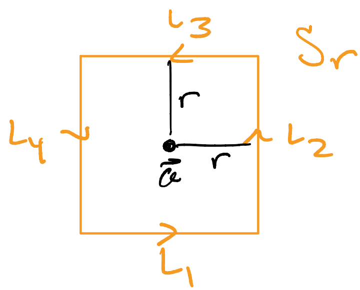
$$g(t^*) = \frac{1}{b-a} \int_a^b g(t)\,dt$$

<u>MVT for Derivatives</u> If g' is continuous on $[a,b]$ then there exists $t^* \in [a,b]$ s.t.

$$g'(t^*) = \frac{g(b) - g(a)}{b-a}$$

Let $\vec{a} = (a_0, a_1) \in \mathcal{U}$

Let S_r be the positively square with sides of length $2r$ centered at $\vec{a}$



$\text{Area}(S_r) = 4r^2$

Label the sides as indicated. Putting a bar over a path means the parameterization is in the wrong direction. Parameterize the sides of S_r as follows:

$$L_1(t) = (a_0 + t,\ a_1 - r) \qquad L_1'(t) = (1, 0)$$

$$L_2(t) = (a_0 + r,\ a_1 + t) \qquad L_2'(t) = (0, 1)$$

$$\overline{L_3}(t) = (a_0 + t,\ a_1 + r) \qquad L_3'(t) = (1, 0)$$

$$\overline{L_4}(t) = (a_0 - r,\ a_1 + t) \qquad L_4'(t) = (0, 1)$$

for $t \in [-r, r]$.

We also split $\vec{F}$ into two parts

$$\vec{F}(x) = \begin{pmatrix} M(\vec{x}) \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ N(\vec{x}) \end{pmatrix}$$

Note that $\displaystyle\int_{S_r} \vec{F} \cdot d\vec{s} = \int_{S_r} \begin{pmatrix} M \\ 0 \end{pmatrix} \cdot d\vec{s} + \int_{S_r} \begin{pmatrix} 0 \\ N \end{pmatrix} \cdot d\vec{s}.$

Then

$$\frac{1}{4r^2}\int_{S_r}\begin{pmatrix}M\\0\end{pmatrix}\cdot d\vec{s}=\frac{1}{4r^2}\left(\int_{L_1}\begin{pmatrix}M\\0\end{pmatrix}\cdot d\vec{s}\;-\;\int_{L_3}\begin{pmatrix}M\\0\end{pmatrix}\cdot d\vec{s}\right)$$

wrong orientation

$$=\frac{1}{2r}\cdot\frac{1}{2r}\int_{-r}^{r}M(a_0+t,a_1-r)-M(a_0+t,a_1+r)\,dt$$

Since

$$\begin{pmatrix}M\\0\end{pmatrix}\cdot\begin{pmatrix}0\\1\end{pmatrix}=0$$

By the MVT for integrals, there is a t^* s.t.

$$\overbrace{M(a_0+t^*,a_1-r)-M(a_0+t^*,a_1+r)}^{g(t^*)}$$

$$=\frac{1}{2r}\int_{-r}^{r}\underbrace{M(a_0+t,a_1-r)-M(a_0+t,a_1+r)}_{g(t)}\,dt$$

Thus,

$$\frac{1}{4r^2}\int_{S_r}\begin{pmatrix}M\\0\end{pmatrix}\cdot d\vec{s}=\frac{M(a_0+t^*,a_1-r)-M(a_0+t^*,a_1+r)}{2r}$$

$$=\frac{M(a_0+t^*,a_1-r)-M(a_0+t^*,a_1+r)}{r-(-r)}$$

Now think of this as the variable!

Now we apply MVT for derivatives

By MVT for derivatives, there is $s^* \in [-r, r]$ so that

here is the variable

note the order

$$\frac{\partial}{\partial y}\bigg|_{y=s^*} M(a_0+t^*, a_1+y) = \frac{M(a_0+t^*, a_1+r) - M(a_0+t^*, a_1-r)}{r-(-r)}$$

$$= -\frac{M(a_0+t^*, a_1-r) - M(a_0+t^*, a_1+r)}{2r}$$

Thus,

$$\frac{1}{4r^2}\int_{S_r} \begin{pmatrix} M \\ 0 \end{pmatrix} \cdot d\vec{s} = -\frac{\partial}{\partial y} M\left(a_0+t^*, a_1+s^*\right)$$

as $r \to 0$ since $t^*, s^* \in [-r, r]$ we have $t^* \to 0$, $s^* \to 0$. Recalling that $\frac{\partial}{\partial y} M$ is continuous since $\vec{F}$ is C^1 shows:

$$\lim_{r\to 0^+} \frac{1}{4r^2}\int_{S_r} \begin{pmatrix} M \\ 0 \end{pmatrix} \cdot d\vec{s} = \lim_{r \to 0^+} -\frac{\partial}{\partial y} M(a_0+t^*, a_1+s^*)$$

$$= -\frac{\partial}{\partial y} M(a_0, a_1)$$

$$= -\frac{\partial}{\partial y} M(\vec{a})$$

A similar computation shows

$$\frac{1}{4r^2}\int_{S_r}\begin{pmatrix}0\\N\end{pmatrix}\cdot d\vec{s} = \frac{1}{4r^2}\left(\int_{L_2}\begin{pmatrix}0\\N\end{pmatrix}\cdot d\vec{s} - \int_{\overline{L_4}}\begin{pmatrix}0\\N\end{pmatrix}\cdot d\vec{s}\right)$$

$$= \frac{\partial}{\partial x} N(a_0+s^*, a_1+t^*)$$

for some $s^*, t^* \in [-r, r]$

Thus:

$$\lim_{r\to 0^+}\frac{1}{4r^2}\int_{S_r}\begin{pmatrix}0\\N\end{pmatrix}\cdot d\vec{s} = \frac{\partial}{\partial x}N(\vec{a})$$

Since $\frac{\partial N}{\partial x}$ is continuous.

Putting it together:

$$\lim_{r\to 0^+}\frac{1}{4r^2}\int_{S_r}\vec{F}\cdot d\vec{s} = \lim_{r\to 0^+}\frac{1}{4r^2}\int_{S_r}\begin{pmatrix}M\\0\end{pmatrix}\cdot d\vec{s} + \lim_{r\to 0^+}\frac{1}{4r^2}\int_{S_r}\begin{pmatrix}0\\N\end{pmatrix}\cdot d\vec{s}$$

$$= -\frac{\partial}{\partial y}M(\vec{a}) + \frac{\partial}{\partial x}N(\vec{a}).$$

Finally we note that we used squares. For other shapes, one can either apply Green's theorem or mimic its proof, as we shall see.