Notes on Constructing Potential Functions

The purpose of this document is to provide some additional details for the proof of:

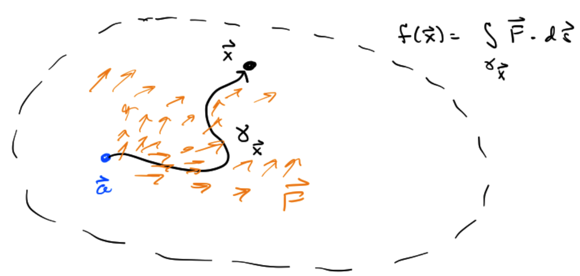
Theorem. Suppose that $U \subset \mathbb{R}^n$ is an open region and that $\mathbf{F}$ is a continuous vector field defined on U such that $\mathbf{F}$ has path independent line integrals. Then $\mathbf{F}$ is conservative. That is, there exists a differentiable scalar field f on U such that $\mathbf{F} = \nabla f$.

To say that U is open means that for every $\mathbf{x} \in U$, there is some positive radius r such that the entire ball of radius r centered at $\mathbf{x}$ is contained in U. This condition can be dropped at the expense of adding some restrictions on the boundary of U.

Proof. For simplicity we assume that U is *path-connected*. This means that for any two points in U there is a piecewise differentiable path in U from one point to the other. If U were not path connected, we would just do the following construction in each piece of U.

We assume that $\mathbf{F}$ has path-independent line integrals. We first define a scalar field f and then we show that $\nabla f = \mathbf{F}$.

Definition: Here is how we define f. Choose a basepoint $\mathbf{a} \in U$. For each $\mathbf{x} \in U$, let $\gamma_{\mathbf{x}}$ be a piecewise differentiable path in U from $\mathbf{a}$ to $\mathbf{x}$. Such a path exists because we are assuming that U is path-connected. Here is a schematic in 2-dimensions:



Define $f(\mathbf{x}) = \int\limits_{\gamma_{\mathbf{x}}} \mathbf{F} \cdot d\mathbf{s}$. Since $\mathbf{F}$ has path-independent line integrals (by assumption), it does not matter what path $\gamma_{\mathbf{x}}$ we pick from $\mathbf{a}$ to $\mathbf{x}$. Any other path (as long as it joins $\mathbf{a}$ to $\mathbf{x}$) will give the same answer for $f(\mathbf{x})$. Thus, $f\colon U \to \mathbb{R}$ is a well-defined function.

Showing it works: We must show that the function f we just defined has the property that $\nabla f = \mathbf{F}$. For simplicity, assuming we are working in two dimensions, so there are real-valued functions M and N so that:

$$\mathbf{F}(\mathbf{x}) = \begin{pmatrix} M(\mathbf{x}) \\ N(\mathbf{x}) \end{pmatrix}$$

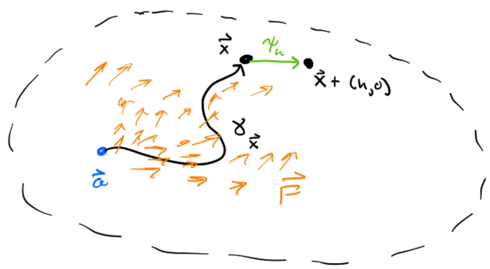
We need to show that $\frac{\partial}{\partial x} f(\mathbf{x}) = M(\mathbf{x})$ and $\frac{\partial}{\partial y} f(\mathbf{x}) = N(\mathbf{x})$. If we are working in more than two dimensions, the proof is similar, we just have more partial derivatives to calculate.

By definition,

$$\frac{\partial}{\partial x} f(\mathbf{x}) = \lim_{h \to 0} \frac{f(\mathbf{x} + (h, 0)) - f(\mathbf{x})}{h}.$$

For simplicity, we will just do the calculation when $h > 0$.

Let $\gamma_{\mathbf{x}}$ be piecewise differentiable path in U from the basepoint $\mathbf{a}$ to the point $\mathbf{x}$. Since U is open, if $|h| > 0$ is small enough, all the points $\mathbf{x} + (t, 0)$ are also in U. Let $\psi_{\mathbf{h}}(t) = \mathbf{x} + (t, 0)$ for $0 \le t \le h$. Here is a depiction of the paths $\gamma_{\mathbf{x}}$ and $\psi_{\mathbf{h}}$:



$$f(\vec{x} + (h,0)) - f(\vec{x}) = \int_{\psi_h} \vec{F} \cdot d\vec{s}$$

Observe that by the definition of f:

$$f(\mathbf{x} + (h,0)) - f(\mathbf{x}) = \int_{\gamma_{\mathbf{x}} \cdot \psi_{\mathbf{h}}} \mathbf{F} \cdot d\mathbf{s} - \int_{\gamma_{\mathbf{x}}} \mathbf{F} \cdot d\mathbf{s}$$

where $\gamma_{\mathbf{x}} \cdot \psi_{\mathbf{h}}$ is the path where we first follow $\gamma_{\mathbf{x}}$ and then follow $\psi_{\mathbf{h}}$. Observe that it is a path from $\mathbf{a}$ to $\mathbf{x} + (h,0)$. Since line integrals can be broken up along segments of path, we have:

$$\int_{\gamma_{\mathbf{x}} \cdot \psi_{\mathbf{h}}} \mathbf{F} \cdot d\mathbf{s} - \int_{\gamma_{\mathbf{x}}} \mathbf{F} \cdot d\mathbf{s} = \int_{\gamma_{\mathbf{x}}} \mathbf{F} \cdot d\mathbf{s} + \int_{\psi_{\mathbf{h}}} \mathbf{F} \cdot d\mathbf{s} - \int_{\gamma_{\mathbf{x}}} \mathbf{F} \cdot d\mathbf{s} = \int_{\psi_{\mathbf{h}}} \mathbf{F} \cdot d\mathbf{s}.$$

And this last integral is something we can compute, since $\psi_{\mathbf{h}}'(t) = (1,0)$. We have:

$$f(\mathbf{x} + (h,0)) - f(\mathbf{x}) = \int_0^h \mathbf{F}(\mathbf{x} + (h,0)) \cdot \psi'(t)\, dt = \int_0^h \begin{pmatrix} M(\mathbf{x} + (h,0)) \\ N(\mathbf{x} + (h,0)) \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \int_0^h M(\mathbf{x} + (h,0)).$$

Thus,

$$\lim_{h \to 0^+} \frac{1}{h}(f(\mathbf{x} + (h,0)) - f(\mathbf{x})) = \lim_{h \to 0^+} \frac{1}{h} \int_0^h M(\mathbf{x} + (h,0)).$$

Notice that the part after the limit is the average value of M on the interval $[0,h]$. By assumption M is continuous, so by the mean value theorem for integrals, there exists $t_h \in [0,h]$ such that

$$\frac{1}{h} \int_0^h M(\mathbf{x} + (h,0)) = M(\mathbf{x} + (t_h, 0)).$$

Thus,

$$\lim_{h \to 0^+} \frac{1}{h}(f(\mathbf{x} + (h,0)) - f(\mathbf{x})) = \lim_{h \to 0^+} M(\mathbf{x} + (t_h,0)) = M(\mathbf{x}).$$

If we then consider the case when $h \to 0^-$, and get the same result, we will have shown:

$$\frac{\partial}{\partial x} f(\mathbf{x}) = M(\mathbf{x})$$

as desired.

Computing $\frac{\partial}{\partial y} f(\mathbf{x})$ is similar, except we use:

$$\psi_{\mathbf{h}}(t) = \mathbf{x} + (0,t).$$

If we do that, we will obtain, via the same argument, $\frac{\partial}{\partial y} f(\mathbf{x}) = N(\mathbf{x})$. That will show:

$$\nabla f(\mathbf{x}) = \begin{pmatrix} \frac{\partial}{\partial x} f(\mathbf{x}) \\ \frac{\partial}{\partial y} f(\mathbf{x}) \end{pmatrix} = \begin{pmatrix} M(\mathbf{x}) \\ N(\mathbf{x}) \end{pmatrix} = \mathbf{F}(\mathbf{x}).$$

□