MA 262	**Vector Calculus**	**Spring 2023**
HW 9	**Stokes Theorem and Gauss' Theorem**	**Due: Fri. 4/14**

These problems are based on your in class work and Sections 7.2 and 7.3 of Colley. Read Section 7.3. Focus on the definition of orientation and compare to the definition given in class. You should memorize the statement of Stokes' Theorem and Gauss' Theorem. Also observe that after these theorems the textbook introduces the "meaning" of curl and divergence, something we've been emphasizing all along.

Some of the problems may look forward to topics we will cover in the future. You should use what you know, think creatively, and not necessarily expect the problems to exactly mimic examples from class or the book. It's your opportunity to practice genuine mathematical thinking!

Please review the course homework policies and don't forget a cover sheet! Whenever you use a computer to calculate or plot you need to say what program/softward you are using. Ideally you will also include all or part of the code you used.

(CR) means the problems are graded on a credit/no credit basis. You are encouraged to check your answers in the back of the book, if possible.

(1) Suppose that $\gamma\colon [a,b] \to \mathbb{R}^3$ and $\psi\colon [a,b] \to \mathbb{R}^3$ are curves. Define the surface $\mathbf{X}$ by:

$$\mathbf{X}(s,t) = (1-s)\gamma(t) + s\psi(t)$$

for all $s \in [0,1]$ and $t \in [a,b]$. This is the surface where we join the curves γ and ψ by line segments. Determine all the points where $\mathbf{X}$ is not smooth.

(2) Prove the following product rule for divergence: Suppose that $f\colon \mathbb{R}^3 \to \mathbb{R}$ is a differentiable scalar fields. Use the partial derivative definition of divergence to show

$$\operatorname{div}(f\nabla f) = (\nabla f)\cdot(\nabla f) + f \operatorname{div}(\nabla f).$$

(3) This problem concerns a basic property of harmonic functions. These are functions that originally arose from the physics of vibrating strings and since have found lots of applications in mathematics and physics.

A scalar field $f\colon \mathbb{R}^3 \to \mathbb{R}$ is **harmonic** if:

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0$$

(this is the sum of the unmixed second partial derivatives. The equation is called Laplace's equation.)

(a) Show that a scalar field is harmonic if and only if $\operatorname{div} \nabla f = 0$. (That is, its gradient field is incompressible.)

(b) Suppose that S is a smooth closed surface in $\mathbb{R}^3$ bounding a region D and that S is oriented with outward pointing normal vector. Suppose that f is a harmonic function and let $\mathbf{G}(\mathbf{x}) = f(\mathbf{x})\nabla f(\mathbf{x})$. Use Gauss' theorem to show that:

$$\iint_S f\nabla f \cdot d\mathbf{S} = \iint_S \mathbf{G}\cdot d\mathbf{S} = \iiint_D ||\nabla f||^2.$$

(Hint: What's the divergence of $f\nabla f$? Use the previous problem)

(c) Use the previous part to explain why if S is a smooth closed surface in $\mathbb{R}^3$ bounding a region D and if S has outward pointing normal vector and if f is a harmonic function such that $f(\mathbf{x}) = 0$ for every $\mathbf{x} \in S$, then in fact $f(\mathbf{x}) = 0$ for every $\mathbf{x} \in D$. (That is, if f is equal to 0 on the boundary of the region D then it is equal to 0 everywhere in D.)

(d) Consider the S and D from the previous two parts. Suppose that g and h are harmonic functions such that $g(\mathbf{x}) = h(\mathbf{x})$ for every $\mathbf{x} \in S$. Use the previous part to show that $g(\mathbf{x}) = h(\mathbf{x})$ for every $\mathbf{x} \in D$. (In other words, if two harmonic functions are equal on the boundary then they are equal everywhere.)

(4) (CR) Do Problems 1, 2, 4, 5, 6 from Section 7.3.

(5) Do Problems 11 and 12 from Section 7.3. (These are trickier than the previous ones)

(6) Do problems 16, 17. Remember that area can be found by integrating 1 over a surface and volume can be found by integrating 1 over a 3-dimensional region.