MA 262	**Vector Calculus**	**Spring 2023**
HW 8	**Parameterized Surfaces**	**Due: Fri. 4/7**

These problems are based on your in class work and Sections 7.1 and 7.2 of Colley. You should additionally take time to consolidate your knowledge of conservative vector fields, scalar curl, curl, divergence, Green's theorem.

Some of the problems may look forward to topics we will cover in the future. You should use what you know, think creatively, and not necessarily expect the problems to exactly mimic examples from class or the book. It's your opportunity to practice genuine mathematical thinking!

Please review the course homework policies and don't forget a cover sheet! Whenever you use a computer to calculate or plot you need to say what program/softward you are using. Ideally you will also include all or part of the code you used.

(CR) means the problems are graded on a credit/no credit basis. You are encouraged to check your answers in the back of the book, if possible.

The problems from Colley on the most recent material are at the end of this problem set.

(1) Determine if the following vector fields are or are not conservative on the indicated region U. For each give a reason. You may be able to find more than one reason for your answer, in which case some are easier to find than others. Take the opportunity to review everything you know about conservative vector fields.

(a) $\mathbf{F}(x,y) = \begin{pmatrix} x\cos(y) \\ y\sin(x) \end{pmatrix}$; $U = \mathbb{R}^2$

(b) $\mathbf{F}(x,y) = \begin{pmatrix} x\cos(y) \\ y\sin(y) \end{pmatrix}$; $U = \mathbb{R}^2$

(c) $\mathbf{F}(x,y) = \begin{pmatrix} y^2 \\ 0 \end{pmatrix}$; $U = \mathbb{R}^2$

(d) $\mathbf{F}(x,y) = \begin{pmatrix} 0 \\ y^2 \end{pmatrix}$; $U = \mathbb{R}^2$

(e) $\mathbf{F}(x,y) = \frac{1}{||(x,y)||}\big(x,y\big)$; $U = \mathbb{R}^2 \setminus \mathbf{0}$

(f) $\mathbf{F}(x,y) = \frac{1}{||(x,y)||^2}\big(x,y\big)$; $U = \mathbb{R}^2 \setminus \mathbf{0}$

(g) $\mathbf{F}(x,y) = \frac{1}{||(x,y)||}\big(-y,x\big)$; $U = \mathbb{R}^2 \setminus \mathbf{0}$

(h) $\mathbf{F}(x,y) = \frac{1}{||(x,y)||^2}\big(-y,x\big)$; $U = \mathbb{R}^2 \setminus \mathbf{0}$

(i) $\mathbf{F}(x,y) = \frac{1}{||(x,y)||^2}\big(-y,x\big)$; $U = \{(x,y) \in \mathbb{R}^2 : y > 0\}$

(2) This problem builds on some problems from the previous two assignments. The goal is to give an example of how we can completely classify irrotational vector fields in terms of the number of "holes" in the domain.

Suppose that $\mathbf{a}_1$, $\mathbf{a}_2$, …, $\mathbf{a}_n$ are distinct points in $\mathbb{R}^2$. Call them **holes**.
Let $U = \mathbb{R}^2 \setminus \{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\}$ (this is the region consisting of all points in the plane except for these holes.)

The point of this problem is to prove the following theorem:

Theorem (Souped up Poincaré Theorem)**.** There exist C^1 vector fields $\mathbf{F}_1, \mathbf{F}_2, \ldots, \mathbf{F}_n$ on U, each having scalar curl equal to 0, such that whenever $\mathbf{G}$ is a C^1 vector field on U with $\operatorname{scurl} \mathbf{G}$ equal to 0, there exist constants $c_1, \ldots, c_n$ and a C^1 scalar field h such that

$$\mathbf{G} = c_1\mathbf{F}_1 + c_2\mathbf{F}_2 + \cdots + c_n\mathbf{F}_n + \nabla h$$

(An aside for those who have had linear algebra: the C^1 vector fields on U with scalar curl equal to 0 form a vector space. This theorem shows that up to the addition of a conservative vector field, the dimension of this vector field is at most n (the number of holes). The vector fields $\mathbf{F}_1, \ldots, \mathbf{F}_n$ are, in fact, also linearly independent and so the dimension is exactly equal to n. This means that the purely algebraic notion of dimension is closely connected to the topology of the space.)

We now begin the proof. Your solution for this problem should contain the entire write up of the proof, including both the parts I provide and the parts that I tell you to provide.

Proof. Let $\mathbf{F}_0(x, y) = \frac{1}{||(x,y)||^2}\begin{pmatrix} -y \\ x \end{pmatrix}$. Let

$$\begin{aligned}
\mathbf{F}_1 &= \mathbf{F}_0(\mathbf{x} - \mathbf{a}_1) \\
\mathbf{F}_2 &= \mathbf{F}_0(\mathbf{x} - \mathbf{a}_2 \\
&\vdots \\
\mathbf{F}_n &= \mathbf{F}_0(\mathbf{x} - \mathbf{a}_n)
\end{aligned}$$

In what follows, consider only positive values of r.

(a) For the purposes of illustrating your work, choose 3 specific points $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$, draw a picture of U and for each of $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ draw a picture of a circle of radius r, with r smaller than half the minimum distance between holes, centered at the point and oriented counter-clockwise. As you go along, you should repeat or augment this picture to help understand what's being asked.

(b) (Extra-credit) For the choice of holes that you made above, use a computer to plot the vector field $\mathbf{F}_1+\mathbf{F}_2+\mathbf{F}_3$ as well as circles of small radius r centered at each of the holes. You may want to adjust your choices of $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_3$ to get a decent picture. For the remaining problems you should work more generally (so in the remaining work, don't assume that $n = 3$ and don't assume that the $\mathbf{a}_i$ are exactly the points you picked.)

(c) Use your previous homework to explain why each $\mathbf{F}_i$ has $\operatorname{scurl} \mathbf{F}_i = 0$.

(d) Let $C_i(r)$ be the circle of radius r centered at $\mathbf{a}_i$ and oriented counter-clockwise. Show that $\int\limits_{C_i(r)} \mathbf{F}_i \cdot d\mathbf{s} = 2\pi$. (Hint: Show that the integral of $\mathbf{F}_0$ around a circle of radius r centered at $(0,0)$ is equal to 2π and then explain why translating both $\mathbf{F}_0$ to $\mathbf{F}_i$ and translating the circle to C_i won't change the result of the integral.)

(e) Suppose that C is a simple closed curve in U, oriented counterclockwise. Explain why for each i, it is the case that:

- If C encloses $\mathbf{a}_i$ then $\int\limits_C \mathbf{F}_i \cdot d\mathbf{s} = 2\pi$
- If C does not enclose $\mathbf{a}_i$, then $\int\limits_C \mathbf{F}_i \cdot d\mathbf{s} = 0$

(Hint: Choose r to be at most half the minimum distance between any two of the holes and small enough so that all the circles $C_i(r)$ are enclosed by C. Apply Green's theorem to the region D bounded by C and the circles $C_i(r)$, noting that each $C_i(r)$ has the wrong orientation for using Green's theorem.)

(f) Suppose that $c_1, c_2, \ldots, c_n$ are numbers, and that C is any simple closed curve in the plane. For each i, let

$$\epsilon_i = \begin{cases} 0 & \text{if } C \text{ encloses } \mathbf{a}_i \\ 1 & \text{if } C \text{ does not enclose } \mathbf{a}_i \end{cases}.$$

Show that

$$\int\limits_C (c_1\mathbf{F}_1 + c_2\mathbf{F}_2 + \cdots + c_n\mathbf{F}_n) \cdot d\mathbf{s} = 2\pi(c_1\epsilon_1 + c_2\epsilon_2 + \cdots c_n\epsilon_n)$$

Now suppose that $\mathbf{G}$ is some unknown C^1 vector field on U with $\operatorname{scurl} \mathbf{G} = 0$. We want to show that $\mathbf{G}$ is the result of adding a conservative vector field to multiples of our known vector fields. Define $k_i = \int\limits_{C_i} \mathbf{G} \cdot d\mathbf{s}$.

(g) Suppose that C is any simple closed curve in U oriented counter-clockwise. Let ϵ_i equal 1 if C encloses $\mathbf{a}_i$ and 0 if it does not. Use Green's theorem to show that

$$\int\limits_C \mathbf{G} \cdot d\mathbf{s} = k_1\epsilon_1 + k_2\epsilon_2 + \cdots + k_n\epsilon_n.$$

(Hint: This is nearly identical to what you did above.)

(h) Let $c_i = \frac{k}{2\pi}$. Define

$$\mathbf{H} = \mathbf{G} - (c_1\mathbf{F}_1 + c_2\mathbf{F}_2 + \cdots + c_n\mathbf{F}_n).$$

Show that if C is any simple closed curve in U that is oriented counter-clockwise, then

$$\int\limits_C \mathbf{H} \cdot d\mathbf{s} = 0.$$

(i) Conclude that $\mathbf{H}$ is conservative and so there exists a potential function h such that $\mathbf{H} = \nabla h$.

(j) Do a minuscule amount of algebra to complete the proof of the theorem.

□

(3) (CR) From Colley Section 7.1 do problems: 1, 2, 5, 12, 13, 14

(4) (CR) From Colley 7.2 do problems: 1, 2, 3, 5, 7