MA 262	Vector Calculus	Spring 2023
HW 7	Green's Theorem	Due: Fri. 3/31

These problems are based on your in class work and Section 6.2 and 6.3's "Criterion for conservative vector fields").

Some of the problems may look forward to topics we will cover in the future. You should use what you know, think creatively, and not necessarily expect the problems to exactly mimic examples from class or the book. It's your opportunity to practice genuine mathematical thinking!

Please review the course homework policies and don't forget a cover sheet! Whenever you use a computer to calculate or plot you need to say what program/softward you are using. Ideally you will also include all or part of the code you used.

(CR) means the problems are graded on a credit/no credit basis. You are encouraged to check your answers in the back of the book, if possible.

(1) (CR) From Colley Section 6.2 do problems: 1, 2, 3, 4, 6, 8, 9, 23

(2) For each of the following statements (discussed in class) give an explanation of why it is true. Your explanation for each should be at least several sentences long and should draw on themes of the class:

(a) Green's Theorem: If $D \subset \mathbb{R}^2$ is a compact region with piecewise C^1 boundary ∂D oriented so that D is on the left, and if $\mathbf{F}$ is a C^1 vector field on D, then

$$\iint_D \operatorname{scurl} \mathbf{F}\, dA = \int_{\partial D} \mathbf{F} \cdot d\mathbf{s}$$

(b) Planar Divergence Theorem: If $D \subset \mathbb{R}^2$ is a compact region with piecewise C^1 boundary ∂D oriented so that D is on the left, and if $\mathbf{F}$ is a C^1 vector field on D, then

$$\iint_D \operatorname{div} \mathbf{F}\, dA = \int_{\partial D} \mathbf{F} \cdot \mathbf{n}\, ds$$

(c) Poincaré's Theorem: If $U \subset \mathbb{R}^2$ is an open simply connected region and if $\mathbf{F}$ is a C^1 vector field on U such that $\operatorname{scurl} \mathbf{F}(x, y) = 0$ for every $(x, y) \in U$ then $\mathbf{F}$ is a conservative on U.

(d) If f is a C^1 scalar field on an open region $U \subset \mathbb{R}^2$, then $\operatorname{scurl} \nabla f = 0$

(e) If $\mathbf{F}$ is a C^1 vector field on an open region $U \subset \mathbb{R}^3$ then $\operatorname{div} \operatorname{curl} \mathbf{F} = 0$.

(f) If $\mathbf{F}$ and $\mathbf{G}$ are conservative vector fields on an open region $U \subset \mathbb{R}^n$, then for any real numbers k, ℓ the vector field $k\mathbf{F} + \ell\mathbf{G}$ defined on U by:

$$(k\mathbf{F} + \ell\mathbf{G})(\mathbf{x}) = k\mathbf{F}(\mathbf{x}) + \ell\mathbf{G}(\mathbf{x})$$

is conservative.

(g) If $\mathbf{F}$ and $\mathbf{G}$ are C^1 vector fields on an open region $U \subset \mathbb{R}^3$, then for any real numbers k, ℓ:

$$\operatorname{curl}(k\mathbf{F} + \ell\mathbf{G}) = k \operatorname{curl} \mathbf{F} + \ell \operatorname{curl} \mathbf{G}$$

(Hint: Although you can do this by cranking through lots of partial derivatives, it is much simpler to use the integral definition of curl.)

(3) Let $\mathbf{F_a}(x,y) = \frac{1}{(x-a_0)^2+(y-a_1)^2}\begin{pmatrix} -(y-a_1) \\ x-a_0 \end{pmatrix}$ where $\mathbf{a} = (a_0, a_1)$. In a previous HW you showed that $\operatorname{scurl} \mathbf{F_0}(x,y) = 0$ for every $(x,y) \neq (0,0)$ but that $\mathbf{F_0}$ is not conservative. The goal of this problem (and one on the next assignment) is to give an example of how we can completely classify all non-conservative irrotational vector fields in terms of the number of "holes" in the domain.

(a) Show that $\operatorname{scurl} \mathbf{F_a}(\mathbf{x}) = 0$ for every $\mathbf{x} \neq \mathbf{a}$. (Hint: Note that $\mathbf{F_a}$ is just a translation of $\mathbf{F_0}$ and you already did this for $\mathbf{F_0}$. How does translation affect scalar curl?)

(b) Let $\mathbb{R}^2 \setminus \mathbf{a}$ be all of the plane except for $\mathbf{a}$ (so it is a plane with a "hole"). Show that $\mathbf{F_a}$ is not conservative on $\mathbb{R}^2 - \mathbf{a}$. (Hint: You already did this when $\mathbf{a} = \mathbf{0}$. You could either adapt that argument or show that translating won't change whether or not a vector field is conservative.)

(c) Let $\mathbf{G}$ be some unknown C^1 vector field on $\mathbb{R}^2 \setminus \mathbf{a}$ with the property that $\operatorname{scurl} \mathbf{G}(\mathbf{x}) = 0$ for every $\mathbf{x} \neq \mathbf{a}$.

(i) Use Green's theorem to show that if C is a simple closed curve in $\mathbb{R}^2$ that does not contain $\mathbf{a}$ either on the curve or inside the curve, then $\int_C \mathbf{F} \cdot d\mathbf{s} = 0$.

(ii) Let γ be any simple closed curve oriented counter clockwise and enclosing $\mathbf{a}$. Define $k = \int_\gamma \mathbf{G} \cdot d\mathbf{s}$. Use Green's theorem to explain why we get the same answer for k, no matter what such curve γ we choose. (We do not however know what the number k is, since we weren't told $\mathbf{G}$.)

(iii) Define

$$\mathbf{H}(\mathbf{x}) = \mathbf{G}(\mathbf{x}) - \frac{k}{2\pi}\mathbf{F_a}(\mathbf{x})$$

for every $\mathbf{x} \neq \mathbf{a}$. Show that if C is *any* simple closed curve not containing $\mathbf{a}$ then

$$\int_C \mathbf{H} \cdot d\mathbf{s} = 0.$$

(Consider two possibilities: either C encloses $\mathbf{a}$ or it does not and for each use the previously established properties of $\mathbf{F_a}$ and $\mathbf{G}$.)

(iv) Appeal to one of our important theorems to conclude that $\mathbf{H}$ is conservative on $\mathbb{R}^2 \setminus \mathbf{a}$. That is, there exists a scalar field h such that $\mathbf{H} = \nabla h$.

(v) Conclude that there is a constant $k \in \mathbb{R}$ such that:

$$\mathbf{G} = \frac{k}{2\pi}\mathbf{F_a} + \nabla h$$

That is, our unknown irrotational vector field $\mathbf{G}$ is actually just a scaled version of our vector field $\mathbf{F_a}$ plus a gradient field. Also note that this means that one this planar region with one hole, up to the addition of conservative vector fields, there is one-dimensions worth of irrotational vector fields. (This dimension is the number k in the formula above, since k could range over the whole number line.)