MA 262	Vector Calculus	Spring 2023
HW 6	Path Independence and Curl	Due: Fri. 3/17

These problems are based on your in class work and Sections 6.3 (skipping the section "Criterion for conservative vector fields") and 3.4 of the textbook (which you should read in that order). Doing this homework in advance will help you study for the exam. Some of the problems may look forward to topics we will cover in the future. You should use what you know, think creatively, and not necessarily expect the problems to exactly mimic examples from class or the book. It's your opportunity to practice genuine mathematical thinking!

Please review the course homework policies and don't forget a cover sheet! Whenever you use a computer to calculate or plot you need to say what program/softward you are using. Ideally you will also include all or part of the code you used.

(CR) means the problems are graded on a credit/no credit basis. You are encouraged to check your answers in the back of the book, if possible.

(1) Review the Change of Variables theorem for integration. This problem gives you an example that seems to contradict it. Your task is to figure out why it doesn't actually contradict the theorem and to explain where the apparent contradiction is coming from.

Let $D^* = [0, 1] \times [0, 2\pi]$ be considered as a rectangle in the uv-plane. Let

$$D = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$$

be the unit disc in the xy-plane. Define $T\colon D^* \to D$ by $T(u, v) = (u\cos(2v), u\sin(2v))$. Let $f(x, y) = x^2 + y^2$.

(a) Compute $\iint\limits_D f \; dx \; dy$, perhaps by converting to polar coordinates.

(b) Compute $\iint\limits_{D^*} f(T(u, v))|\det DT(u, v)| \; du \; dv$

(c) Note that the previous two answers are not equal and that this seems to contradict the Change of Variables Theorem. Explain where the apparent contradiction comes from and why this example does not actually contradict the Theorem. (Hint: Think about how T maps D^* onto D.)

(2) For the purposes of this problem, a particle either has charge $+1$ or a charge of -1. One version of Coulomb's Law says that (with appropriate choices of units) if a particle of charge ϵ_1 is at point $\mathbf{a} \in \mathbb{R}^3$ and if a particle of charge ϵ_2 is at a point $\mathbf{x} \in \mathbb{R}^3$, then the force of attraction of $\mathbf{b}$ on the point at $\mathbf{x}$ is given by:

$$\mathbf{F_a}(\mathbf{x}) = \frac{\epsilon_1 \epsilon_2}{||\mathbf{x} - \mathbf{a}||^3}(\mathbf{x} - \mathbf{a})$$

(a) Show that $\mathbf{F_a}(\mathbf{x})$ is a conservative vector field and find a potential function for it. (Hint: adapt what we did for gravitational force to this problem.)

(b) Suppose there are positively charged particles, one each on $(-2, 0)$ and $(2, 0)$. Use the principle of super-position to write down the equation of a vector field $\mathbf{F}$ representing the combined force of attraction of those two particles on a negatively charged particle p at a generic point $\mathbf{x}$.

(c) Use a property of gradients to show that $\mathbf{F}$ is also conservative and to find a potential function for it.

(d) Suppose that the particle p moves from the point $(1,0)$ to the point $(-1,0)$ via the half circle $\gamma(t) = (\cos t, \sin t)$ for $t \in [0,\pi]$. Compute the work done. (Hint: there's an easy way and there's a hard way....)

(e) Suppose that the particle p instead moves along the line segment $\alpha(t) = (t,0)$ for $t \in [-1,1]$. Compute the work done. (Hint: There's a hard way, an easy way, and an extremely easy way.)

(3) (CR) From Section 6.3, do problems 1, 2, 3, 18

(4) Let $f(x,y) = \sin(xy) + x - y$. Use a computer to plot both the scalar field (as a density plot/heat map) and its gradient vector field on the same plot.

(5) In class we showed that if $\mathbf{F}$ is a vector field in a path-connected region U has path independent line integrals, then it is conservative. We did this by choosing a basepoint $\mathbf{a} \in U$, and defining the potential function f by choosing a path $\gamma_{\mathbf{x}}$ from $\mathbf{a}$ to $\mathbf{x}$ and defining $f(\mathbf{x}) = \int_{\gamma_{\mathbf{x}}} \mathbf{F} \cdot d\mathbf{s}$. If we change the definition of f by replacing $\mathbf{a}$ with a different basepoint $\mathbf{b} \in U$, how does that change the function f? Why? (Hint: consider the integral of $\mathbf{F}$ over a path from $\mathbf{a}$ to $\mathbf{b}$)

(6) Suppose that $\mathbf{F}$ is a vector field on a region U and that $\alpha\colon [0,1] \to U$ and $\beta\colon [0,1] \to U$ are curves. Define

$$\gamma(t) = \begin{cases} \alpha(t) & 0 \le t \le 1 \\ \beta(t-1) & 1 \le t \le 2 \end{cases}$$

for $t \in [0,2]$.

(a) Explain why γ is a parameterization of the curve obtained by first following the curve α and then following the curve β.

(b) Explain why:

$$\int_{\gamma} \mathbf{F} \cdot d\mathbf{s} = \int_{\alpha} \mathbf{F} \cdot d\mathbf{s} + \int_{\beta} \mathbf{F} \cdot d\mathbf{s}$$

This and similar calculations justify our propensity to not try to find a parameterization of a complicated shape (like a square!) but rather just use parameterizations of its individual pieces.

(7) Let $\mathbf{F}(x,y) = \begin{pmatrix} y \\ -x^2 \end{pmatrix}$. For this problem we will want to calculat the integral of $\mathbf{F}$ over all possible squares with horizontal and vertical sides and notice some patterns. You will want to sketch pictures as you work through this.

(a) Let $S(r,\mathbf{a})$ be the square centered at $\mathbf{a}$ with width and height $2r > 0$ and oriented counterclockwise. That is, if $\mathbf{a} = (a_0, a_1)$ it is the square with corners $(a_0 \pm r, a_1)$ and $(a_0, a_1 \pm r)$. Find separate parameterizations of each of the fours sides of $S(r,\mathbf{a})$. Be sure your parameterizations make the square oriented counter-clockwise. You will want to make your parameterizations as simple as you can.

(b) For a generic $\mathbf{a}$ and r, compute $\int_{S(r,\mathbf{a})} \mathbf{F} \cdot d\mathbf{s}$. Your answer may involve a_0, a_1, r. Also compute $\lim_{r \to 0^+} \int_{S(r,\mathbf{a})} \mathbf{F} \cdot d\mathbf{s}$.

(c) Note that the area of $S(r, \mathbf{a}) = 4r^2$. Compute:

$$\lim_{r \to 0^+} \frac{1}{4r^2} \int\limits_{S(r,\mathbf{a})} \mathbf{F} \cdot d\mathbf{s}.$$

(d) Show that your answer to the previous problem is equal to $\frac{\partial}{\partial x}(-x^2) - \frac{\partial}{\partial y}(y)$ evaluated at the point $x = a_0$ and $y = a_1$.

(8) Let $U \subset \mathbb{R}^2$ be a region whose boundary is given by piecewise C^2 curves. Suppose that γ is an oriented curve in U. Let $\mathbf{n}(t)$ be the unit normal vector to obtained by rotating the unit tangent vector to γ $\pi/2$ radians clockwise. The **flux** of a vector field $\mathbf{F}$ across γ is the line integral:

$$\text{flux}(\mathbf{F}, \gamma) = \int\limits_{\gamma} \mathbf{F} \cdot \mathbf{n}\, ds$$

(a) Use properties of the dot product and the line integral to explain why this can be considered the "total amount" of the vector field passing across the curve γ.

(b) For the following vector fields $\mathbf{F}$ and curves γ compute the flux of $\mathbf{F}$ across γ. Be sure to sketch the vector field and curve as part of your work, but you don't need to turn the pictures in.

(i) $\mathbf{F}(x, y) = (x, y)$; $\gamma(t) = (r\cos t, r\sin t)$ for $t \in [0, 2\pi]$

(ii) $\mathbf{F}(x, y) = (x, y)$; γ is the square with corners at $(\pm 1, \pm 1)$ and oriented counter-clockwise. (You'll have to define a piecewise parameterization of γ or else break the computations into four pieces.)

(iii) $\mathbf{F}(x, y) = (-y, x)$; $\gamma(t) = (r\cos t, r\sin t)$ for $t \in [0, 2\pi]$

(iv) $\mathbf{F}(x, y) = (-y, x)$; γ is the square with corners at $(\pm 1, \pm 1)$ and oriented counter-clockwise.

(c) Explain why reversing the orientation of a curve reverses its flux.

(9) This problem introduces an important example that we'll refer to repeatedly. On the region $U = \mathbb{R}^2 \setminus \mathbf{0}$ (that is, the plane except for the origin) let

$$\mathbf{F}_0(x, y) = \frac{1}{x^2 + y^2} \begin{pmatrix} -y \\ x \end{pmatrix}.$$

(a) Use a computer to plot the vector field. Sketch in some flow lines.

(b) Let C_r be the circle of radius r centered at the origin and oriented counter-clockwise. Find a parameterization of C_r so that C_r is a flow line.

(c) Compute $\int\limits_{C_r} \mathbf{F}_0 \cdot d\mathbf{s}$. (Hint: Use the fact that C_r is a flow line.)

(d) Use your answer to the previous part to show that $\mathbf{F}_0$ is not conservative.

(e) Use the partial derivative definition of scalar curl (or curl) to show that the scalar curl of $\mathbf{F}_0$ is equal to 0. This means the vector field is irrotational.

One other fact: (We'll prove this later) The vector field $\mathbf{F}_0$ has the property that if you integrate it around a closed curve that *does not* contain the origin, then the result of the integral is 0.

(10) From Section 3.4: do problems 1, 2, 3, 7, 8, 9 using the formulas from that section.