MA 262	**Vector Calculus**	**Spring 2023**
HW 4	**Changing Coordinate Systems**	**Due: Fri. 3/3**

These problems are based on your in class work and Sections 2.5, 5.5, 3.3, 6.1 of the textbook (which you should read). Some of them may look forward to topics we will cover in the future. You should use what you know, think creatively, and not necessarily expect the problems to exactly mimic examples from class or the book. It's your opportunity to practice genuine mathematical thinking!

Please review the course homework policies and don't forget a cover sheet! Whenever you use a computer to calculate or plot you need to say what program/softward you are using. Ideally you will also include all or part of the code you used.

(CR) means the problems are graded on a credit/no credit basis. You are encouraged to check your answers in the back of the book, if possible.

(1) (CR) Perform (by hand) the following matrix computations or say why it is not possible. Then check your work using a computer (include your computer computations as part of your solutions.)

(a)

$$\begin{pmatrix} 2 & 1 & -3 \end{pmatrix} \begin{pmatrix} 4 \\ 0 \\ 6 \end{pmatrix}.$$

(b)

$$\begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ 0 \\ 6 \end{pmatrix}.$$

(c)

$$\begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix} \begin{pmatrix} 4 \\ 0 \\ 6 \end{pmatrix}.$$

(d)

$$\begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 6 \\ -3 \end{pmatrix}$$

(e)

$$\begin{pmatrix} 6 \\ -3 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix}$$

(f)

$$\begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 6 & 0 \\ -3 & 1 \end{pmatrix}$$

(g)

$$\begin{pmatrix} 2 & -1 & 8 \\ 1 & 2 & \pi \end{pmatrix} \begin{pmatrix} 6 & 0 \\ -3 & 1 \\ 4 & -1 \end{pmatrix}$$

(h)

$$\begin{pmatrix} 2 & -1 & 8 \\ 1 & 2 & \pi \end{pmatrix} \begin{pmatrix} 6 & 0 & 1 \\ -3 & 1 & 0 \end{pmatrix}$$

(2) (CR) Colley Section 2.3 Problems: 26, 29, 31, 32

(3) The purpose of this problem is to help you understand why the Chain rule is true. Suppose that $G\colon \mathbb{R}^m \to \mathbb{R}^n$ are differentiable at $\mathbf{a} \in \mathbb{R}^m$ and that $F\colon \mathbb{R}^n \to \mathbb{R}^p$ is differentiable at $G(\mathbf{a}) \in \mathbb{R}^n$.

(a) Write down the formula for the linear approximation to G based at $\mathbf{a}$. Call it $L(\mathbf{x})$.

(b) Write down the formula for the linear approximation to F based at $G(\mathbf{a})$. Call it $M(\mathbf{x})$.

(c) Plug your formula for $L(\mathbf{x})$ into your formula for $M(\mathbf{x})$ to get a formula for $M(L(\mathbf{x}))$.

(d) Rearrange your formula so that it is of the form $M(L\mathbf{x})) = A(\mathbf{x} - \mathbf{a}) + \mathbf{b}$ where A is some matrix (perhaps the product of two other matrices!) and $\mathbf{b}$ is some vector (perhaps expressed as the combination of a bunch of other vectors.)

(e) Here is the "plausability argument" for the chain rule. Read and understand.

proof idea. If $\mathbf{x}$ is close to $\mathbf{a}$, we have $G(\mathbf{x}) \approx L(\mathbf{x})$. When $\mathbf{y}$ is close to $G(\mathbf{a})$ we have $F(\mathbf{y}) \approx M(\mathbf{y})$. Since G is continuous, when $\mathbf{x}$ is very close to $\mathbf{a}$, $G(\mathbf{x})$ is close to $G(\mathbf{a})$. Thus, when $\mathbf{x}$ is very close to $\mathbf{a}$ we have

$$F(G(\mathbf{x})) \approx M(L(\mathbf{x})) = A(\mathbf{x} - \mathbf{a}) + \mathbf{b}$$

Thus, we have a linear approximation to $F \circ G$ based at $\mathbf{a}$. The matrix A is must then be the derivative (because that's what derivatives are!) so $D(F \circ G)(\mathbf{a}) = A$. And if you found the right expression for A, then you have just "proven" the chain rule. The only thing missing is a rigorous handling of all the approximations and that's best left for another course. $\square$

(4) Colley Section 5.5 Problems: 1, 4, 8, 10 (this one is hard – use a change of variables!), 13, 15, 23

(5) A clown is running through a mirror funhouse following the path

$$\gamma(t) = (t + 5\cos(2\pi t), 1 + \sin(2\pi t)$$

for $t \geq 0$. Consider the floor of the funhouse as the uv-plane. Consider the mirrored ceiling of the funhouse to be the xy-plane. The mirror on the ceiling is shaped in such a way that the point (x, y) shows the reflection of the point $(ue^{3v}, u^2 - 4v^2)$.

(a) What is the velocity and speed of the clown at time $t = 3/4$?

(b) Let $T(u, v) = (ue^{3v}, u^2 - 4v^2)$. Find DT and use it to find the velocity and speed of the clown's reflection at time $t = 3/4$.

(c) At time $t = 3/4$ how does the speed and velocity of the clown's reflection compare to the speed and velocity of the clown?

(d) On the floor of the funhouse is a square tile with coordinates $[0, \frac{1}{8}] \times [0, \frac{1}{8}]$. What is the area of its reflection on the ceiling? (Hint: Use the change of coordinates theorem for double integrals). You may give a numerical approximation as your answer.

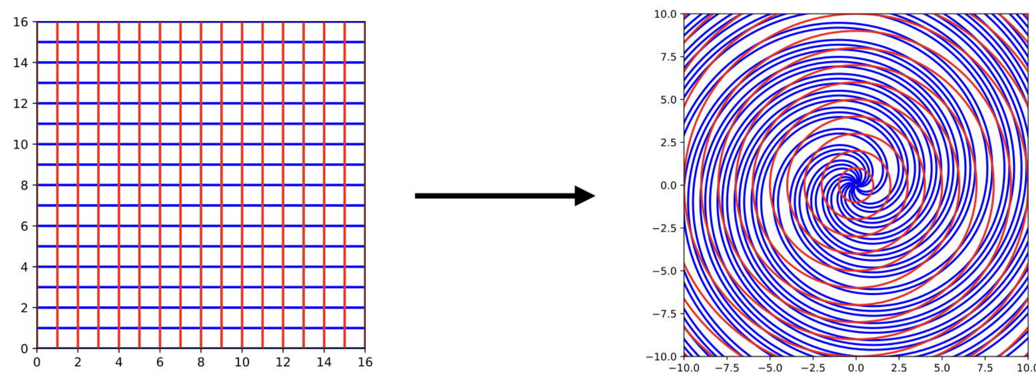
(6) (Based on 2.5, problem 10) A bird flies along the helical curve $x = 2\cos t$, $y = 2\sin t$, $z = 3t$. It suddenly encounters a weather front so that the barometric pressure is varying rather wildly from point to point as $P(x, y, z) = 6x^2yz$.

(a) Use the chain rule to determine how the pressure is changing at $t = \pi/4$.
(b) Check your result from (a) by direct substitution.
(c) What is the approximate pressure at $t = \pi/4 + 0.01$ minute?

(7) Consider the coordinate transformation $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ defined by:

$$(x, y) = T(u, v) = (u\cos(u + v), u\sin(u + v)).$$

The effect of this transformation on a grid is shown in the image.



(a) Compute the derivative of T, as well as its determinant.

(b) At what points (u, v) is T singular? orientation reversing? orientation-preserving?

(c) At what points is T expanding? at what points is it contracting?

(d) Consider the square $R = [0, \pi/4] \times [0, \pi/4]$ in the $u-v$ plane. Compute the image $T(R)$ of R under the transformation, by following these steps and using an online graphing program to help, as needed:

(i) Let $u_-(t) = (t, 0)$ for $t \in [0, \pi/4]$ and $u_+(t) = (\pi/4 - t, \pi/4)$ for $t \in [0, \pi/4]$. These curves trace out the bottom of the square R from left to right and the top of R from right to left. Compute and draw the curves $T(u_-(t))$ and $T(u_+(t))$.
(ii) Let $v_+(t) = (\pi/4, t)$ for $t \in [0, \pi/4]$ and $v_-(t) = (0, \pi/4 - t)$ for $t \in [0, \pi/4]$. These curves trace out the right and left sides of R from bottom to top and top to bottom, respectively. Compute and draw the curves $T(v_+(t))$ and $T(v_-(t))$.
(iii) Shade in the area trapped by the four curves, $T(u_\pm(t))$ and $T(v_\pm(t))$. This is $T(R)$.

(e) Compute the area of $T(R)$ using a double integral and the change of coordinates theorem.