MA 262	**Vector Calculus**	**Spring 2023**
HW 3	**Curves and Coordinates**	**Due: Fri. 2/24**

This is the second part part of HW #2. You should turn your solutions to both parts in together as one assignment. It was split into two parts only because of the logistics surrounding my travel.

These problems are based on your in class work and Sections 3.1, 3.2 (through p. 210), and 5.5 of the textbook (which you should read). Some of them may look forward to topics we will cover in the future. You should use what you know, think creatively, and not necessarily expect the problems to exactly mimic examples from class or the book. It's your opportunity to practice genuine mathematical thinking!

Please review the course homework policies and don't forget a cover sheet! Whenever you use a computer to calculate or plot you need to say what program/softward you are using. Ideally you will also include all or part of the code you used.

(CR) means the problems are graded on a credit/no credit basis. You are encouraged to check your answers in the back of the book, if possible.

(1) (CR) From Colley Section 3.1: Problems 30 and 31. For Problem 30a, show that $||\mathbf{x}(t)||^2 = 1$ for all t. For Problem 30b, show that $\mathbf{x}\cdot\dot{\mathbf{x}} = 0$. For 30c, use the product rule for differentiable parameterized curves:

$$\frac{d}{dt}\mathbf{a}(t) \cdot \mathbf{b}(t) = \mathbf{a}(t) \cdot \dot{\mathbf{b}}(t) + \dot{\mathbf{a}}(t) \cdot \mathbf{b}(t).$$

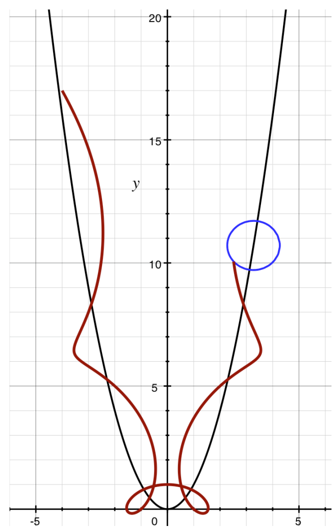
(2) (CR) From Colley Section 3.2: Problems 1, 3, 7, 10, 14, 17 (curvature only), 18 (curvature only)

(3) A parameterization of the graph of the sine function is given by $\gamma(t) = (t, \sin(t))$. Write down an integral equal to its length for $t \in [0, 8\pi]$ and use a computer to find a numerical approximation to the length.

(4) Each of the following parts gives you two parameterized paths γ and ψ. Determine if ψ is a reparameterization of γ and, if so, what the change of coordinates function is and if it is orientation-preserving or reversing. Give a reason for each answer.

(a) $\gamma(t) = (t\cos t, e^t)$ for $t \in [-2\pi, 2\pi]$; $\psi(t) = (4\pi t\cos(4\pi t), e^{4\pi t})$ for $t \in [-1/2, 1/2]$.
(b) $\gamma(t) = (t\cos t, e^t)$ for $t \in [-2\pi, 2\pi]$; $\psi(t) = (4\pi t\cos(4\pi t), e^{4\pi t})$ for $t \in [-2\pi, 2\pi]$.
(c) $\gamma(t) = (t, t\ln(t), 4t)$ for $t \in [1, 10e]$; $\psi(t) = (-t^3/3, -t^3\ln(-t^3/3)/3, -4t^3/3)$ for $t \in [-\sqrt[3]{30e}, -\sqrt[3]{3}]$.

(5) Suppose that a circle of radius 1 is rolling down a hill such that the center of the circle is always on the graph of the parabola $y = x^2$. The circle rolls in such that the center of the circle is at the point (t, t^2) at time t and it completes 1 clockwise rotation every 2 seconds. At time $t = 0$, the center of the circle is at the point $(0, 0)$. Let P be the point on the circle directly above the center of the circle at time $t = 0$. **Find** the parameterization of the path $\mathbf{x}(t)$ taken by the point P as the circle rolls down the parabola. See the image below for a depiction. **Bonus:** Use a computer to plot the path you find (you do not have to plot the parabola and circle unless you want to.) Turn in a still of your animation and a screenshot of what you wrote to get it to work.



(6) In Ptolemaic astronomy, the motion of another planet around the earth was described using rotating circles whose centers were also rotating circles, whose centers were also rotating circles, and so on. (This is called the "epicycle on a deferent" description.) Apparently, it provided a highly accurate description of planetary motion. It can also be used to describe the teacup ride at Disneyland.

This problem gives you an indication of how you might work out aspects of the planet's motion, in a simplified setting. Suppose that a point $p(t)$ is rotating at ω_1 revolutions per second counter-clockwise around a point $q(t)$ which is itself rotating clockwise at ω_2 revolutions per second around a point $r(t)$ that is rotating counter-clockwise around the origin at ω_3 revolutions per second. The distance from $p(t)$ to $q(t)$ is always 1. The distance from $q(t)$ to $r(t)$ is always 3. The distance from $r(t)$ to the origin is always 5. At time 0, the centers of all three circles, as well as the point p are on the positive x-axis and the x-coordinates are, in increasing order: $r(0)$, $q(0)$, $p(0)$

(a) Find the coefficients k, m so that $r(t) = \begin{pmatrix} m\cos(kt) \\ m\sin(kt) \end{pmatrix}$ for all t.
(b) In coordinates centered at $r(t)$ find $q(t)$.
(c) In coordinates centered at $q(t)$ find $p(t)$.
(d) In coordinates centered at the origin, find $p(t)$.

(7) Sometimes in graphic design, the designer wants to have two curves a little bit apart from each other that together trace out a thick path, as in Figure 1. One way to do this, is to start with one curve $\gamma(t)$ and then come up with another curve $\psi(t)$ that we get by moving

$\gamma(t)$ perpendicular (aka orthogonal) to itself some distance δ. This problem works out the formula for $\psi(t)$, given the formula for $\gamma(t)$. Assume that $\gamma(t) = (x(t), y(t))$ is differentiable.

(a) Given a vector $\begin{pmatrix} a \\ b \end{pmatrix}$, find a vector orthogonal to it. (Hint: Find values c, d, expressed in terms of a and b, so that

$$\begin{pmatrix} a \\ b \end{pmatrix} \cdot \begin{pmatrix} c \\ d \end{pmatrix} = 0.$$

(b) Use the previous part to find a formula (in terms of $\dot{x}, \dot{y}$) for a vector $\mathbf{n}(t)$ orthogonal to $\dot{\gamma}(t)$.

(c) Suppose that we want ψ to be a distance δ from γ. Find a formula for ψ. Your answer will involve γ, $\mathbf{n}$, and δ.

(d) Let $\gamma(t) = \begin{pmatrix} t \\ t\sin(t) \end{pmatrix}$. Let $\delta = 0.2$. Find a formula for $\psi(t)$ and plot both γ and ψ on the same axes.

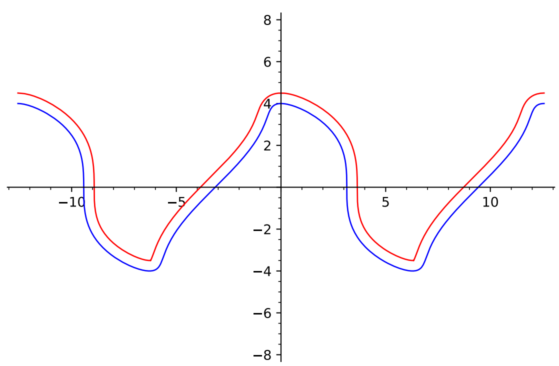


FIGURE 1. An example of parallel curves

(8) If $\gamma\colon [a, b] \to \mathbb{R}^3$ is a differentiable path, there is a natural coordinate system at each point along the path called the Frenet frame. It's the coordinate system used by aircraft, drones, birds, etc. Since we are discussing a path in 3-dimensions, we specify three vectors $\mathbf{T}(t)$, $\mathbf{N}(t)$, and $\mathbf{B}(t)$, corresponding to "straight ahead", "the inside of the curve", and "twisting into the 3rd dimension". Their names are the: unit tangent vector, unit normal, and unit binormal. The formulas are:

$$\begin{aligned} \mathbf{T}(t) &= \tfrac{1}{||\dot{\gamma}(t)||}\dot{\gamma}(t) \\ \mathbf{N}(t) &= \tfrac{1}{||\dot{\mathbf{T}}(t)||}\dot{\mathbf{T}}(t) \\ \mathbf{B}(t) &= \mathbf{T}(t) \times \mathbf{N}(t) \end{aligned}$$

(a) As a warm-up, compute the unit tangent, unit normal, and unit binormal vectors for the helix $\gamma(t) = (\cos(t), \sin(t), t)$.

(b) Explain why $\mathbf{T}$ and $\mathbf{N}$ are perpendicular. (Hint: one of the assigned problems from Colley will help!)

(c) Explain why $\mathbf{B}$ is perpendicular to both $\mathbf{T}$ and $\mathbf{N}$. (Hint: review the properties of the cross product)

(d) Explain why $\mathbf{N}$ always points to the inside of the curve. (Hint: what does the derivative of $\mathbf{T}$ measure?)

(9) (CR) Colley Section 2.3, Problems 27, 29

(10) (CR) Colley Section 2.5, Problems 19, 23