MA 262	Vector Calculus	Spring 2021
HW 2a	**Double Integrals**	**Due: Fri. 2/17**

This is the first part part of HW #2. The second part will be posted on Monday.

These problems are based on your in class work and Chapter 5 of the textbook (which you should read). Some of them may look forward to topics we will cover in the future. You should use what you know, think creatively, and not necessarily expect the problems to exactly mimic examples from class or the book. It's your opportunity to practice genuine mathematical thinking!

Please review the course homework policies and don't forget a cover sheet! Whenever you use a computer to calculate or plot you need to say what program/softward you are using to do the plot. Ideally you will also include all or part of the code you used.

(CR) means the problems are graded on a credit/no credit basis. You are encouraged to check your answers in the back of the book, if possible.

(1) (CR) From Colley Section 5.1, do Exercises 1, 3, 6 without using technology

(2) (CR) From Colley Section 5.1, do Exercises 7, 8, 9

(3) Write a paragraph, in your own words explaining the relationship between a double integral and an iterated integral, including what the difference is.

(4) Write a paragraph, in your own words, explaining when one should attempt to express a double integral over a non-rectangular region as an iterated integral with respect to $dx\,dy$ as opposed to $dy\,dx$.

(5) Consider the function $f(x,y) = \frac{1}{(x^2+y^2)^{3/2}}$. Let R be the annular region between the circles having the equations $x^2+y^2=2$ and $x^2+y^2=4$. (Draw a picture of R!)

  (a) Explain why the integral $\iint\limits_R f\,dA$ is equal to $4\iint\limits_U f\,dA$ where U is just the portion of R contained in the first quadrant of the plane.

  (b) Write $\iint\limits_U f\,dA$ as one or more iterated integrals.

  (c) Use software to solve your iterated integral(s) and then compute (a numerical approximation to) $\iint\limits_R f\,dA$.

(6) Suppose that $R \subset \mathbb{R}^2$ is a bounded region, bounded by piecewise smooth curves (this means that the boundary is made up of finitely many smooth curves joined to together at their endpoints, for example a polygon). Use Riemann sums to explain why $\iint\limits_R 1\,dA$ is equal to the area of R.

(7) The point of this problem is to establish a connection between double integrals and average value. For example, one would use integrals to compute the average temperature of some object or the average pressure of the air in a chamber.

Suppose that R is subdivided by horizontal and vertical lines into an $p \times q$ grid of smaller rectangles. See Figure 1. The left sides of the rectangle are two of the vertical lines and the distance between two adjacent vertical lines is always the same number: Δx. Similarly, the

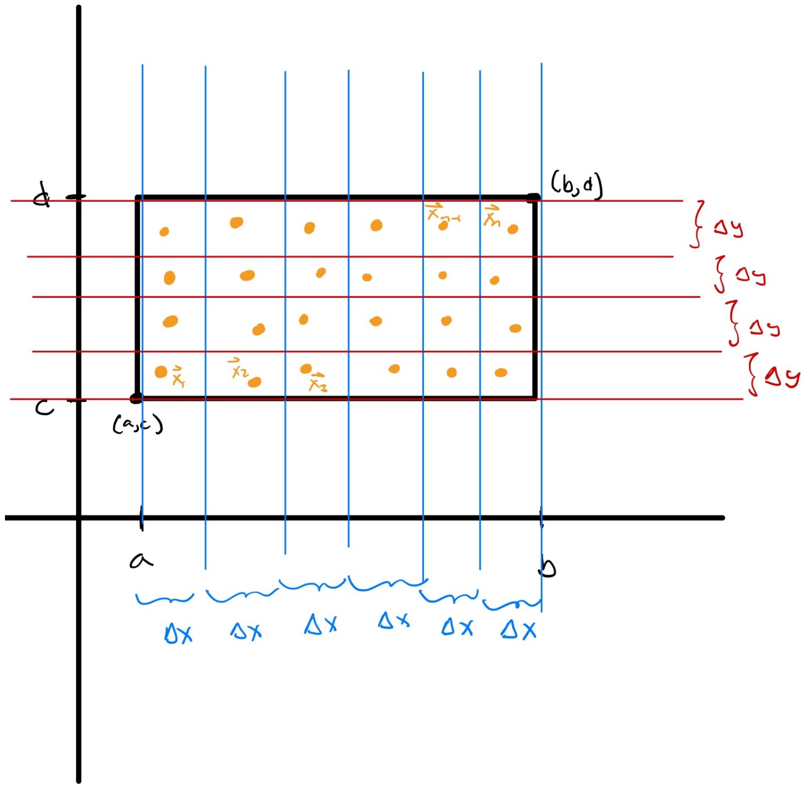

FIGURE 1. A grid of subrectangles, each containing a sample point.

top and bottom of the rectangle are two of the horizontal lines and the distance between two adjacent horizontal lines is always the same number: Δy. Let $n = pq$.

(a) Find an expression for n in terms of area(R), Δx, and Δy.

(b) Assume that each of the sample points $\mathbf{x}_1^*, \ldots, \mathbf{x}_n^*$ is in its own subrectangle, as in Figure 1. Rewrite your formula for the average value of f on the n points solely in terms of area(R), Δx, and Δy.

(c) Explain why it makes sense to *define* the average value of f over the whole rectangle R (and not just the sample points) to be the limit as $n \to \infty$ of the formula from the previous part.

(d) Use the preceding part to explain why the integral formula for average value given at the outset of this problem is a sensible definition for the average value of f on R.

(e) Why is it important that our grid of smaller rectangles be an $p \times q$ array of rectangles all the same size? For instance, why shouldn't we just divide R into n vertical strips, each of width Δy or why shouldn't we pick all our sample points near the lower left corner of R?

(f) Suppose that the rectangle R represents a thin sheet of metal such that the temperature of the metal at the coordinate (x, y) is $x^2 + 2y^2$. Find the average temperature of the metal. (Your answer will involve a, b, c, and d.)

(8) A **curve** is a function $\gamma\colon [a,b] \to \mathbb{R}^n$ where $[a,b]$ is an interval. For example, $\gamma\colon [0,1] \to \mathbb{R}^2$ defined by $\gamma(t) = \begin{pmatrix} t \\ t^2 \end{pmatrix}$ for $t \in [0,1]$ is a curve. We could also write $\gamma(t) = (t, t^2)$ for $t \in [0,1]$. Use software to plot the following curves. Print your plot and include as part of your homework. Take the time to think about why the given formula produces the picture it does.

(a) $\gamma(t) = (\cos(t), \sin(t))$ for $t \in [0, 2\pi]$

(b) $\gamma(t) = (t\cos(t), t\sin(t))$ for $t \in [0, 4\pi]$

(c) $\gamma(t) = (1-t)\begin{pmatrix} 3 \\ -2 \end{pmatrix} + t\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ for $t \in [0,1]$.

(d) $\gamma(t) = (t^3, t^3)$ for $t \in [0,3]$.

(9) If $\gamma(t) = (x(t), y(t))$ is a curve in $\mathbb{R}^2$, its **derivative** is $\dot{\gamma}(t) = (\dot{x}(t), \dot{y}(t))$ (where $\dot{x}$ denotes the derivative of the function x with respect to t). For example, if $\gamma(t) = (t^2, \cos(t))$ then $\dot{\gamma}(t) = (2t, -\sin(t))$. Find the derivatives of the following curves:

(a) $\gamma(t) = (t^2, \cos(t^2))$ for $t \in [0, \pi]$.

(b) $\gamma(t) = (1-t)\begin{pmatrix} 3 \\ -2 \end{pmatrix} + t\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ for $t \in [0,1]$.

(10) Suppose that $\gamma(t) = (x(t), y(t))$ is a differentiable curve in $\mathbb{R}^2$.

(a) The function $x(t)$ is a function $x\colon \mathbb{R} \to \mathbb{R}$, like what you studied in Calc 1. It represents the x-coordinate of the curve at time t. Using your Calc 1 knowledge, explain what $\dot{x}(t)$ measures. Similarly, explain what $\dot{y}(t)$ measures.

(b) Suppose that $\gamma(t)$ has the property that both $\dot{x}$ and $\dot{y}$ are constant. Explain why this means that γ is a straight line.

(c) On the other hand, by casting your eye back over the problems you've already done, find an example of a function where $\dot{x}$ and $\dot{y}$ are not constant, but γ is still a straight line. Explain this phenomenon.

(d) Show that if $\mathbf{a}$ and $\mathbf{b}$ are vectors, then

$$\gamma(t) = (1-t)\mathbf{a} + t\mathbf{b}$$

is the line segment from $\mathbf{a}$ to $\mathbf{b}$.