MA 262	**Vector Calculus**	**Spring 2023**
HW 10	**Divergence in Cylindrical Coordinates**	**Due: Fri. 5/5**

These problems are based on your in class work and Problem 29 of Section 7.3 of Colley. Please carefully read the notes "Divergence in Cylindrical Coordinates" posted to the course webpage. These problems are all (CR). They are intended to help you think carefully through aspects of what we did in class, as well as the more detailed version appearing in the notes.

(1) In class we discussed the challenge of writing the vector field $\mathbf{F}(x, y, z) = (x, y, z)$ in cylindrical coordinates. The first part of the notes walks you through the process of finding the formula, beginning with the relevant formula for polar coordinates. Thinking about that process:

(a) Recall that when we base a coordinate system at a point $\mathbf{p}$ in $\mathbb{R}^2$, we call the the **tangent space** at that point. (Technically: the tangent space at $\mathbf{p}$ is all vectors based at $\mathbf{p}$.) In the usual rectangular coordinates for the tangent space, the axes are horizontal and vertical. Why does it make sense to label them dx and dy? (Hint: a vector in the tangent space with respect to those coordinates tells us how far to move horizontally and vertically from $\mathbf{p}$. What does the slope of the vector represent? What is the connection to the tangent line to a curve going through that point?) (For an additional resource if you want one, look at the section on "Differentials" in any Calc I textbook.)

(b) When the point p is at some angle θ (measured from the positive x-axis) and some radius r (measured from the origin in $\mathbb{R}^2$), why does it make sense to line up the dr axis with the vector based at $\mathbf{p}$ and pointing in the direction $\mathbf{p}$? (Hint: in what direction should we move if we move at an angle 0 from p?)

(c) When the point p is at some angle θ (measured from the positive x-axis) and some radius r (measured from the origin in $\mathbb{R}^2$), why does it make sense to make the $d\theta$ axis perpendicular to the dr axis? (Hint: If r stays constant and θ changes, we move in a circle – what's the relationship between a tangent vector to the circle and the radius of the circle?)

(d) In the notes, the previous observations lead to an expression of a vector field in polar coordinates. To get the expression for a vector field in cylindrical coordinates, we simply use the expression for polar coordinates but at each height z. Why is it that adding a z-component to the vector field doesn't change the angle of movement in the horizontal plane or the distance from the z-axis?

(2) The second part of the notes derives the cylindrical expression for divergence. These questions concern that derivation.

(a) Explain why the unit normal vectors to the top and bottom faces of the surface ∂V depicted in the notes point straight up and straight down (respectively) and are represented by the vectors $(0, 0, 1)$ and $(0, 0, -1)$ in cylindrical coordinates.

(b) Explain why the unit normal vectors to the front and back faces of the surface ∂V depicted in the notes points are represented by $(0, 1, 0)$ and $(0, -1, 0)$ in cylindrical coordinates. (Hint: as you start at a point p on one of those faces and then move orthogonally out from the face does the angle you move out depend on the point p? Does the radius or height change when you move perpendicularly out?)

(c) Explain why the unit normal vectors to the left and right faces of the surface ∂V depicted in the notes points are represented by $(1, 0, 0)$ and $(-1, 0, 0)$ in cylindrical coordinates. (Hint: as you start at a point p on one of those faces and then move orthogonally out from the face does the radius you move out depend on the point p? Does the angle or height change when you move perpendicularly out?)

(d) Explain why, in calculating $\iint\limits_{\partial V} \mathbf{F} \cdot d\mathbf{S}$ we can split the integral into integrals of F_z over the top and bottom faces; F_θ over the front and back faces; and F_r over the left and right faces.

(e) Of those calculations the integral of F_z over the top and bottom faces is the easiest, thought they all follow the same pattern. In those integrals the "Mean Value Theorem for Integrals" is used. This is a theorem from Calc 1. State it carefully (as from a Calc 1 textbook) and then say exactly how it is used in our calculation of the integral of F_z over the top and bottom faces. (You might find it helpful to look at the other notes I provided a while ago showing that the integral formula for scalar curl is equal to the partial derivative formula.)

(f) The calculations also use the Mean Value Theorem for Derivatives (also just called "The Mean Value Theorem"). State it carefully (as from a Calc 1 textbook) and then say exactly how it is used in our calculation of the integral of F_z over the top and bottom faces. Be sure to say what the function is and what the interval is that the theorem is being applied to. (You might find it helpful to look at the other notes I provided a while ago showing that the integral formula for scalar curl is equal to the partial derivative formula.)

(g) As we mentioned, the integrals of F_z are the easiest of the three (though not easy!). As an example of one of the harder ones, explain how the Mean Value Theorem for Derivatives is used in the calculation of the integral of F_r over the left and right faces.

(3) Problem 30 in Section 7.3 asks you to work out the formula for divergence in spherical coordinates. You don't have to do it, but write a few sentences sketching the idea of how it would work; comparing it to our derivation of the formula in cylindrical coordinates.