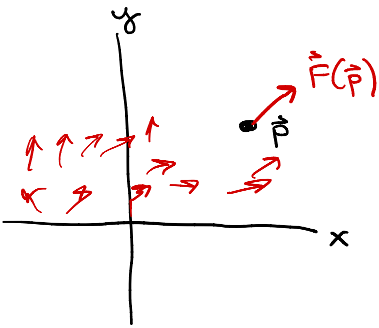
Vector Fields in cylindrical coords

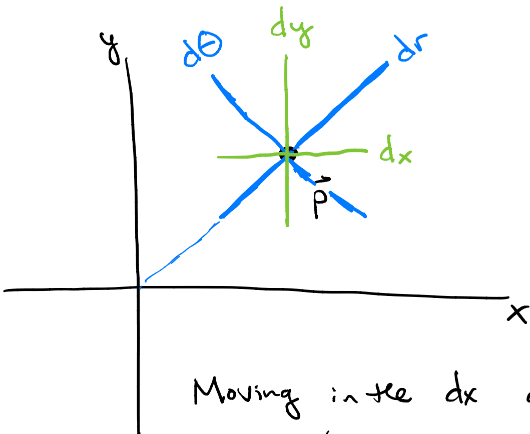
① Consider a 2D vector field in polar coords



If in rectangular coords

$$\vec{F}(x,y) = \begin{pmatrix} M(x,y) \\ N(x,y) \end{pmatrix}$$

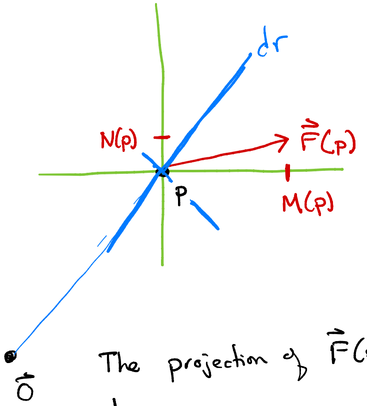
We want to find a corresponding expression when in polar coords



Consider the tangent plane based at p. We observe two coordinate systems one given by dx, dy and one given by dr, dθ

Moving in the dx direction by Δx increases the x-coord. of our point by Δx. Moving in the dr direction by Δr increases distance to origin by Δr.

Thus to figure out the vector field in polar coords:



The projection of $\vec{F}(p)$ onto $dr = \frac{1}{r}\vec{p}$ is given by

$$\frac{F(\vec{p}) \cdot \vec{p}}{\|\vec{p}\|^2}\vec{p} = \frac{F(\vec{p}) \cdot \vec{p}}{r} dr$$

The vector $d\theta$ is orthogonal to dr (and thus to p)

so if $\vec{p} = \begin{pmatrix} x \\ y \end{pmatrix}$ then $d\theta = \frac{1}{r}\begin{pmatrix} -y \\ x \end{pmatrix}$

the projection of $\vec{F}(p)$ onto $d\theta$ axis then

$$\frac{1}{r} F(\vec{p}) \cdot \begin{pmatrix} -y \\ x \end{pmatrix} d\theta$$

$\Rightarrow$ IN polar coords

$$\vec{F}(r,\theta) = \frac{1}{r}\begin{pmatrix} M(x,y)\,x + N(x,y)\,y \\ -M(x,y)\,y + N(x,y)\,x \end{pmatrix}$$

$$= \begin{pmatrix} M(x,y)\cos\theta + N(x,y)\sin\theta \\ -M(x,y)\sin\theta + N(x,y)\cos\theta \end{pmatrix}$$

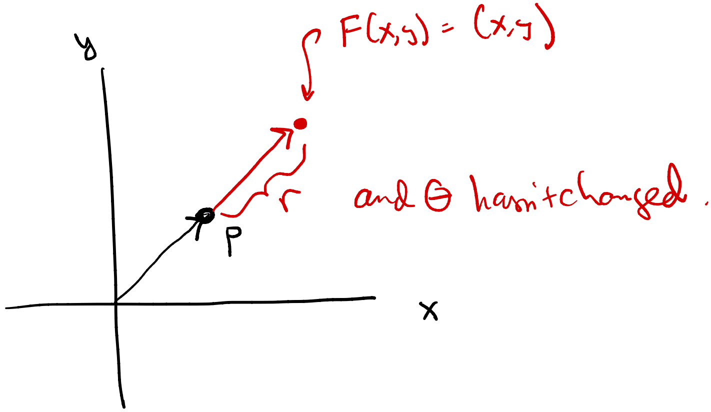
Ex If $\vec{F}(x,y) = \begin{pmatrix} x \\ y \end{pmatrix}$ then in polar:

$$\vec{F}(r,\theta) = \begin{pmatrix} x\cos\theta + y\sin\theta \\ -x\sin\theta + y\cos\theta \end{pmatrix}$$

$$= \begin{pmatrix} r\cos^2\theta + r\sin^2\theta \\ -r\cos\theta\sin\theta + r\sin\theta\cos\theta \end{pmatrix}$$

$$= \begin{pmatrix} r \\ 0 \end{pmatrix}$$

which makes sense:



IN cylindrical coords, the vector field

$$\vec{F}(x,y,z) = \begin{pmatrix} M(x,y,z) \\ N(x,y,z) \\ P(x,y,z) \end{pmatrix}$$

is represented as

$$\vec{F}\begin{pmatrix} r \\ \theta \\ z \end{pmatrix} = \begin{pmatrix} M\cos\theta + N\sin\theta \\ N\cos\theta - M\sin\theta \\ P \end{pmatrix}$$

<u>Ex</u> $F(x,y,z) = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ in polar coords is:

$$\vec{F}(r,\theta,z) = \begin{pmatrix} r\cos^2\theta + r\sin^2\theta \\ r\sin\theta\cos\theta - r\sin\theta\cos\theta \\ z \end{pmatrix} = \begin{pmatrix} r \\ 0 \\ z \end{pmatrix}$$

Theorem In cylindrical coords the divergence of

$$\vec{F}(r,\theta,z) = \begin{pmatrix} F_r(r,\theta,z) \\ F_\theta(r,\theta,z) \\ F_z(r,\theta,z) \end{pmatrix}$$

is

$$\operatorname{div} \vec{F} = \frac{1}{r}\frac{\partial}{\partial r}(r F_r) + \frac{1}{r}\frac{\partial}{\partial \theta} F_\theta + \frac{\partial}{\partial z} F_z$$

Ex If $\vec{F}(x,y,z) = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ divergence is

$$\frac{\partial}{\partial x} x + \frac{\partial}{\partial y} y + \frac{\partial}{\partial z} z = 3$$

In cylindrical coordinates

$$\vec{F}(r,\theta,z) = \begin{pmatrix} r \\ 0 \\ z \end{pmatrix}$$

$$\begin{aligned} \operatorname{div} F &= \frac{1}{r}\frac{\partial}{\partial r}(r^2) + \frac{1}{r}\frac{\partial}{\partial \theta} 0 + \frac{\partial}{\partial z} z \\ &= \frac{1}{r}(2r) + 0 + 1 \\ &= 2+1 \\ &= 3 \end{aligned}$$

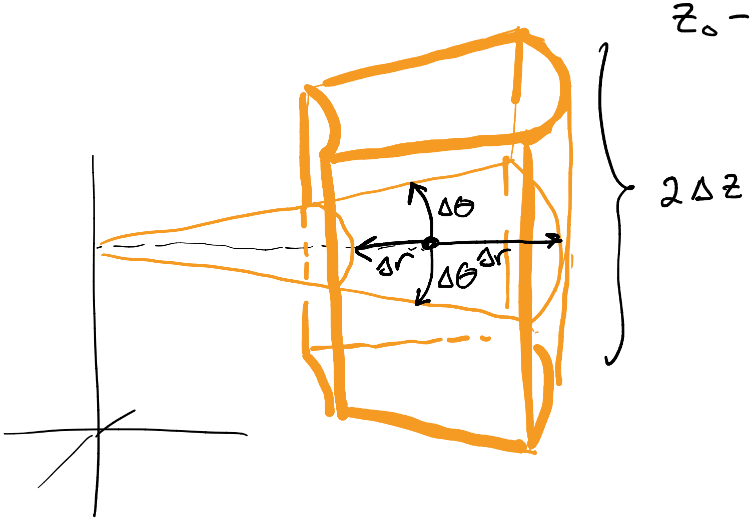
Derivation of cylindrical divergence formula:

Let $\vec{a} = (x_0, y_0, z_0) = (r_0, \theta_0, z_0)$ in $\mathbb{R}^3$

Consider $\Delta r > 0 \quad \Delta\theta > 0 \quad \Delta z > 0$

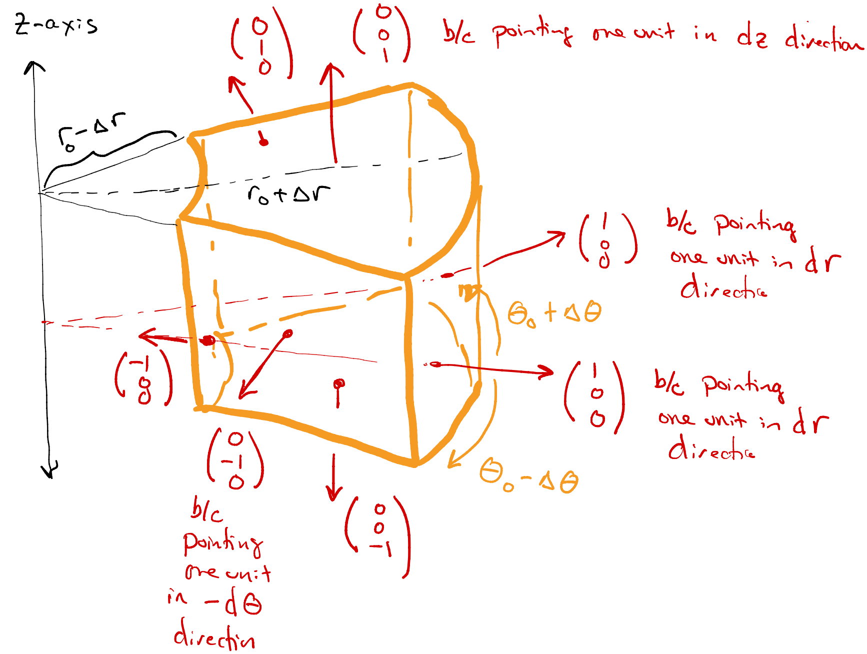
and the 3D region

$$V = \left\{ (r, \theta, z) \;\middle|\; \begin{array}{l} r_0 - \Delta r \le r \le r_0 + \Delta r \\ \theta_0 - \Delta\theta \le \theta \le \theta_0 + \Delta\theta \\ z_0 - \Delta z \le z \le z_0 + \Delta z \end{array} \right\}$$



We give the ∂V outward normals

IN rectangular coords these are difficult to figure out, but in cylindrical coords its easier!



Observe:

$$\iint_{\partial V} \vec{F}\cdot d\vec{S} = \iint_{\text{top}} F_z\, dS - \iint_{\text{bottom}} F_z\, dS$$

$$+ \iint_{\text{right}} F_r\, dS - \iint_{\text{left}} F_r\, dS$$

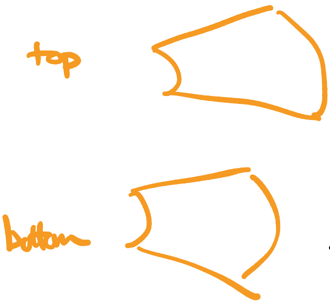
$$+ \iint_{\text{back}} F_\theta\, dS - \iint_{\text{front}} F_\theta\, dS$$

Also $\text{Vol}(V) = \text{Area}(\text{top})\,(2\Delta z)$

$$= \frac{2\Delta\theta}{2\pi}\left(\pi(r_0+\Delta r)^2 - \pi(r_0-\Delta r)^2\right)(2\Delta z)$$

$$= 8\, r_0\, \Delta r\, \Delta\theta\, \Delta z$$

We take the pieces one at a time:



parameterize the top as

$$\begin{pmatrix} r \\ \theta \\ z_0 + \Delta z \end{pmatrix} \quad \text{for} \quad \begin{matrix} r_0 - \Delta r \le r \le r_0 + \Delta r \\ \theta_0 - \Delta\theta \le \theta \le \theta_0 + \Delta\theta \end{matrix}$$

and the bottom as

$$\begin{pmatrix} r \\ \theta \\ z_0 - \Delta z \end{pmatrix}$$

with the same bounds.

then

$$\frac{1}{Vol(V)}\left(\iint_{top} F_z \, dS - \iint_{bottom} F_z \, dS\right)$$

$$= \frac{1}{r_0}\frac{1}{2\Delta z}\frac{1}{Area(top)}\int_{\theta_0-\Delta\theta}^{\theta_0+\Delta\theta}\int_{r_0-\Delta r}^{r_0+\Delta r}\Big(F_z(r,\theta,z_0+\Delta z) - F_z(r,\theta,z_0-\Delta z)\Big)\overbrace{r\,dr\,d\theta}^{\text{in polar}}$$

By the MVT for integrals (applied twice) there is r^*, θ^* so:

$$= \frac{1}{r_0}\frac{1}{2\Delta z}\underbrace{\Big(F_z(r^*,\theta^*,z_0+\Delta z) - F_z(r^*,\theta^*,z_0-\Delta z)\Big)r^*}_{\text{function of } z \text{ evaluated at end pts}}$$

By MVT for derivatives

there is an z^* with $z_0 - \Delta z \le z^* \le z_0 + \Delta z$

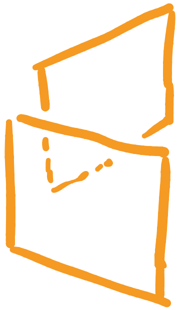
so that

$$= \frac{1}{r_0}\frac{\partial}{\partial z}F_z(r^*,\theta^*,z^*)\,r^* \quad \text{Since } (r^*,\theta^*,z^*) \in V$$

as V shrinks to (r_0, θ_0, z_0), as the partial deriv is continuous,

$$\lim \frac{1}{Vol(V)}\left(\iint_{top} F_z \, dS - \iint_{bottom} F_z \, dS\right) = \frac{\partial}{\partial z}F_z(r_0,\theta_0,z_0)$$

Now we consider the front and back. Parameterize them as



$$S \begin{pmatrix} r \\ \theta_0 + \Delta\theta \\ z \end{pmatrix}$$

$$S \begin{pmatrix} r \\ \theta_0 - \Delta\theta \\ z \end{pmatrix}$$

Then:

$$\frac{1}{\mathrm{Vol}(V)} \left(\iint_{\text{back}} F_\theta \, dS - \iint_{\text{front}} F_\theta \, dS \right) =$$

notice no r b/c $\Delta r \Delta z$ does not depend on r

$$\frac{1}{r_0} \frac{1}{2\Delta\theta} \frac{1}{2\Delta z} \frac{1}{2\Delta r} \int_{z_0-\Delta z}^{z_0+\Delta z} \int_{r_0-\Delta r}^{r_0+\Delta r} F_\theta(r, \theta_0+\Delta\theta, z) - F_\theta(r, \theta_0-\Delta\theta, z) \, dr\,dz$$

By MVT for integrals, there is r^*, z^* so

$$= \frac{1}{r_0} \frac{1}{2\Delta\theta} \Big(\underbrace{F_\theta(r^*, \theta_0+\Delta\theta, z^*) - F_\theta(r^*, \theta_0-\Delta\theta, z^*)}_{\text{function of } \theta \text{ evaluated at endpts of } [\theta_0-\Delta\theta,\ \theta_0+\Delta\theta]} \Big)$$

By MVT for derivatives there is θ^* so

$$= \frac{1}{r_0} \frac{\partial}{\partial\theta} F_\theta(r^*, \theta^*, z^*)$$

Assuming the partial deriv. is continuous this approaches

$$\frac{1}{r} \frac{\partial}{\partial\theta} F_\theta(r, \theta, z) \quad \text{as } V \text{ shrinks to } \vec{a}.$$

Finally we consider the left and right sides

In cylindrical coords they are parameterized as

$$\begin{pmatrix} r_0 - \Delta r \\ \theta \\ z \end{pmatrix} \text{ and } \begin{pmatrix} r_0 + \Delta r \\ \theta \\ z \end{pmatrix}$$

We have

$$(*) \quad \frac{1}{\mathrm{Vol}(V)} \left(\iint_{\text{right}} F_r \, dS - \iint_{\text{left}} F_r \, dS \right)$$

this is from the Jacobian of polar coords

$$= \frac{1}{\mathrm{Vol}(V)} \Bigg(\int_{\theta_0 - \Delta\theta}^{\theta_0 + \Delta\theta} \int_{z_0 - \Delta z}^{z_0 + \Delta z} F_r(r_0 + \Delta r, \theta, z)\,(r_0 + \Delta r)\, dz\, d\theta$$

$$- \int_{\theta_0 - \Delta\theta}^{\theta_0 + \Delta\theta} \int_{z_0 - \Delta z}^{z_0 + \Delta z} F_r(r_0 - \Delta r, \theta, z)\,(r_0 - \Delta r)\, dz\, d\theta \Bigg)$$

$$= \frac{1}{\mathrm{Vol}(V)} \int_{\theta_0 - \Delta\theta}^{\theta_0 + \Delta\theta} \int_{z_0 - \Delta z}^{z_0 + \Delta z} F_r(r_0 + \Delta r, \theta, z)(r_0 + \Delta r) - F_r(r_0 - \Delta r, \theta, z)(r_0 - \Delta r) \; dz\, d\theta$$

MVT Integrals

$$= \frac{1}{r_0} \frac{1}{2\Delta r} \left(F_r(r_0 + \Delta r, \theta^*, z^*)(r_0 + \Delta r) - F_r(r_0 - \Delta r, \theta^*, z^*) \right) \quad \text{for some } \theta^*, z^*$$

$$= \frac{1}{r_0} \frac{\partial}{\partial r}\Big|_{r^*} \left(F_r(r, \theta^*, z^*)\, r \right) \quad \text{for some } r^*$$

as $V \to \vec{a}$, $(r^*, \theta^*, z^*) \to \vec{a}$. As we assume the partial deriv. is continuous,

$$\lim_{V \to \vec{a}} (*) \longrightarrow \frac{1}{r_0} \frac{\partial}{\partial r} F_r(r_0, \theta_0, z_0).$$

∎