Curl

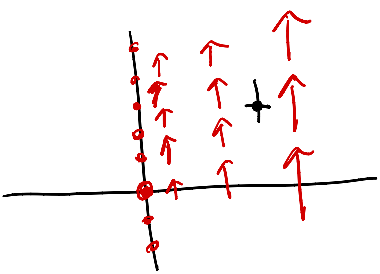
Recall for a C^1 vector field $\vec{F} = \begin{pmatrix} M \\ N \end{pmatrix}$ on $\mathcal{U} \subset \mathbb{R}^2$

$$\lim_{C \to \vec{x}} \frac{1}{\text{Area}(C)} \int_C \vec{F} \cdot d\vec{s} = \frac{\partial}{\partial x} N(\vec{x}) - \frac{\partial}{\partial y} M(\vec{x})$$

currently really just for squares C

(show SquaresCirclesVF)

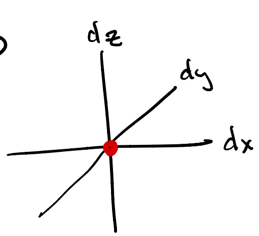
Ex $\quad F(x,y) = \begin{pmatrix} 0 \\ x^2 \end{pmatrix}$ $\quad$ scurl $F(x,y) = -2x$

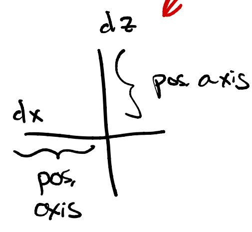


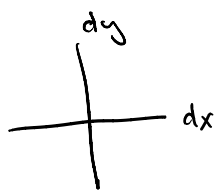
A paddle wheel will spin (no matter how small)

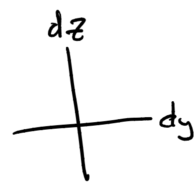
Note LH coordinates

IN 3D

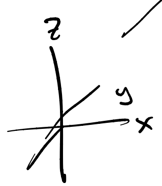








$\frac{\partial}{\partial x} N - \frac{\partial}{\partial y} M \qquad \frac{\partial}{\partial y} P - \frac{\partial}{\partial z} N$

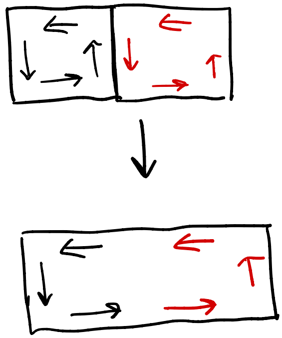


$F = \begin{pmatrix} M \\ N \\ P \end{pmatrix}$ $\qquad -\left(\frac{\partial}{\partial x} P - \frac{\partial}{\partial z} M\right)$

b/c LH

Points in axis of swirling

Observe some interesting properties of circulation



Green's Theorem

Suppose that $D \subset \mathbb{R}^2$ is a compact region with piecewise C^1 boundary, oriented so D is on the left and that $\vec{F} : D \to \mathbb{R}^2$ is a C^1 vector field then

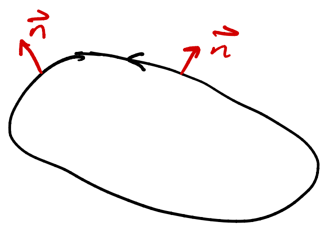
$$\iint_D \operatorname{scurl} \vec{F} \, dA = \int_{\partial D} \vec{F} \cdot d\vec{s}$$

"Adding scalar curl over a region is equivalent to the circulation along the boundary."

Flux

flux of $\vec{F}$ across an oriented curve C

is $\int_C F\cdot\vec{n}\,ds$ $\quad$ $\vec{n}$ obtained by rotating $\vec{T}$ to the right $\pi/2$



Compare to circulation:

$$\int_C \vec{F}\cdot d\vec{s} = \int_a^b F(\gamma(t))\cdot\gamma'(t)\,dt$$
$$= \int_a^b F(\gamma(t))\cdot\underbrace{\left(\frac{\gamma'(t)}{\|\gamma'(t)\|}\right)}_{\text{unit tangent vector}}\|\gamma'(t)\|\,dt$$
$$= \int_C F\cdot\vec{T}\,ds$$

positive flux : more leaving region than entering

divergence $\quad \operatorname{div}\vec{F}(\vec{a}) = \lim_{C\to\vec{a}} \frac{1}{\text{Area}(C)}\int_C \vec{F}\cdot\vec{n}\,ds$

for C^1 v.f. $\longrightarrow$ $= \dfrac{\partial M}{\partial x} + \dfrac{\partial N}{\partial y}$

$\operatorname{curl}(\nabla f) = 0$

$\operatorname{div}(\operatorname{curl}\vec{F}) = 0$

$\operatorname{div}\vec{F} = 0$ means "incompressible"

Planar Divergence Theorem

(Same hypotheses as Green)

$$\iint_D \operatorname{div} \vec{F} \, dA = \int_{\partial D} \vec{F} \cdot \vec{n} \, dA$$