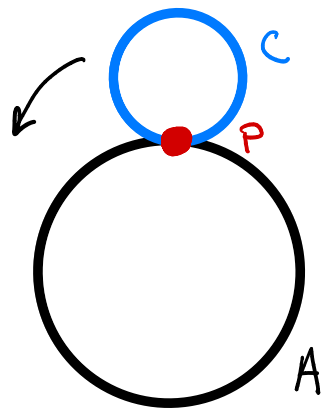
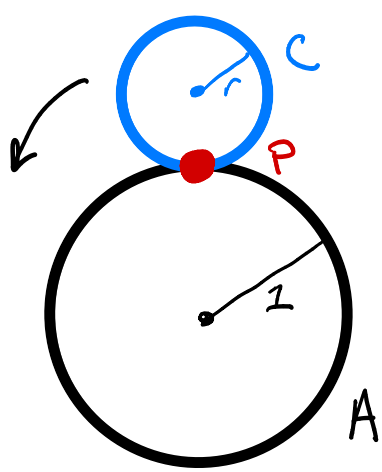
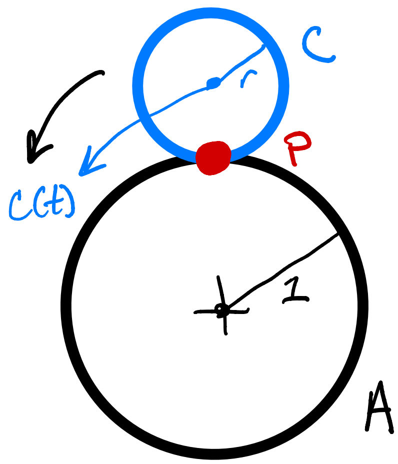
Notes on the cardiod

Find a parameterization of a point p on a circle C that rolls clockwise around a circle A without sliding on slipping.



Assume that A starts on the very top of C and that p is initially the point of tangency. Also assume A has radius 1 and is centered at the origin and that C has radius r





Let $c(t)$ denote the center of C at time t. It is rotating counterclockwise in a circle of radius $(1+r)$. Thus

$$c(t) = (1+r)\begin{pmatrix}\cos(\omega t + \alpha)\\ \sin(\omega t + \alpha)\end{pmatrix}$$

The number ω accounts for the rate of rotation. We will have to figure that out. The number α controls the starting position of C. Since C starts at the top of A, $\alpha = \pi/2$.

In coordinates centered at $c(t)$, the point P is traveling in a circle, counterclockwise so its position is given by

$$p(t) = \begin{pmatrix} r\cos(\hat{\omega} t + \beta)\\ r\sin(\hat{\omega} t + \beta)\end{pmatrix}_{c(t)}$$

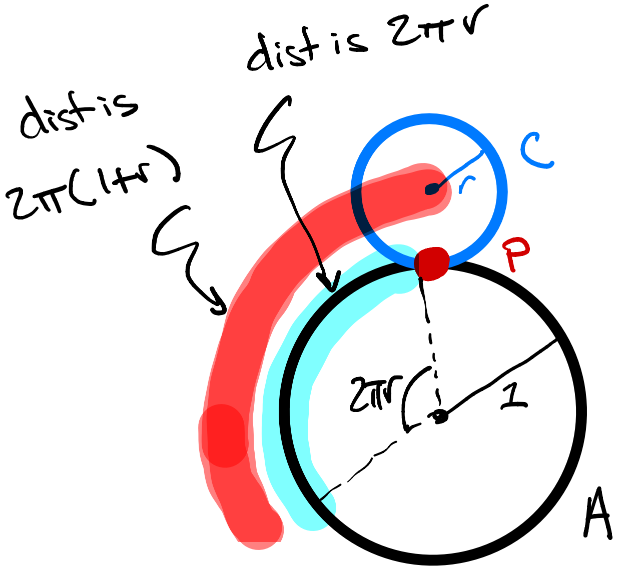
coords based at $c(t)$

Where $\hat{\omega}$ is the rate of rotation and β is starting position. Since the point starts at the bottom of C $\beta = -\frac{\pi}{2}$. We will have to figure out $\hat{\omega}$.

Thus, in standard coordinates

$$p(t) = (1+r)\begin{pmatrix}\cos(\omega t + \pi/2)\\ \sin(\omega t + \pi/2)\end{pmatrix} + r\begin{pmatrix}\cos(\hat{\omega} t - \pi/2)\\ \sin(\hat{\omega} t - \pi/2)\end{pmatrix}$$

Now to figure out ω and $\hat{\omega}$.



consider unwrapping C as it rolls. After 1 revolution it will travel $2\pi r$ along A. That means it traces out an angle of $2\pi r$

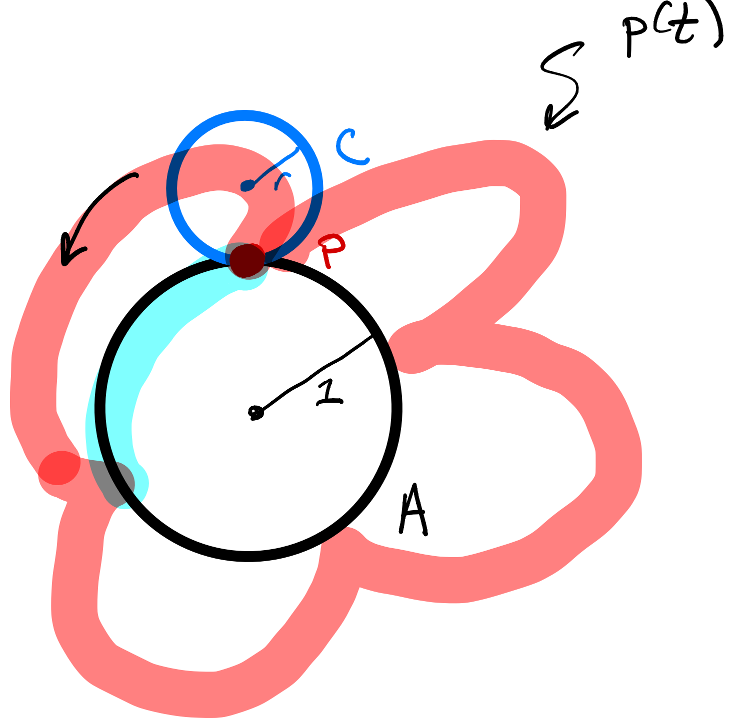
It does that in $\frac{2\pi}{\omega}$ sec. since $\cos\left(\omega\left(\frac{2\pi}{\omega}\right)\right) = \cos(2\pi)$

The center on the other hand travels $(1+r)\,2\pi$. We want it to cover that distance in $\frac{2\pi}{\omega}$ seconds so the circle doesn't slip or slide. Thus, we need

$$\frac{2\pi}{\omega} = (1+r)\cdot 2\pi$$

Thus $\omega = \frac{1}{1+r}$.

Similarly:



From the point of view of the center of C, the point of tangency rotates around C at the same rate as P. In one revolution the point of tangency covers a dist of $2\pi r$ (imagine unwrapping C as it rolls) it does so in $\frac{2\pi}{\hat{\omega}}$ sec. We need it to do so in $2\pi r$ sec, else there is slipping or sliding.

Thus $\frac{2\pi}{\hat{\omega}} = 2\pi r$, so $\hat{\omega} = \frac{1}{r}$.

Hence

$$p(t) = (1+r)\begin{pmatrix}\cos\left(\frac{1}{1+r}t+\frac{\pi}{2}\right)\\ \sin\left(\frac{1}{1+r}t+\frac{\pi}{2}\right)\end{pmatrix} + r\begin{pmatrix}\cos\left(\frac{1}{r}t-\frac{\pi}{2}\right)\\ \sin\left(\frac{1}{r}t-\frac{\pi}{2}\right)\end{pmatrix}.$$

$\square$