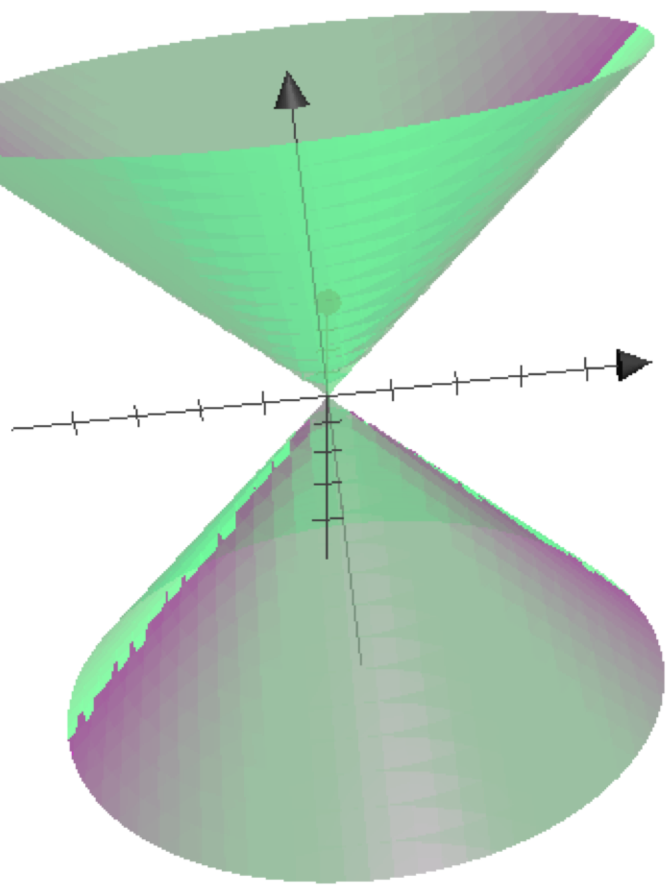
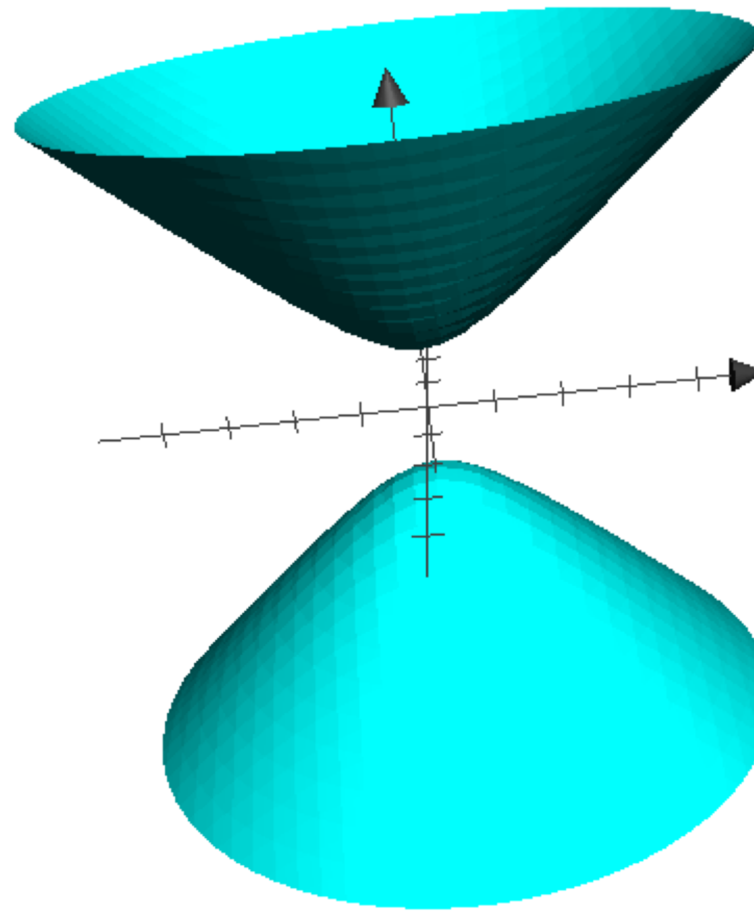
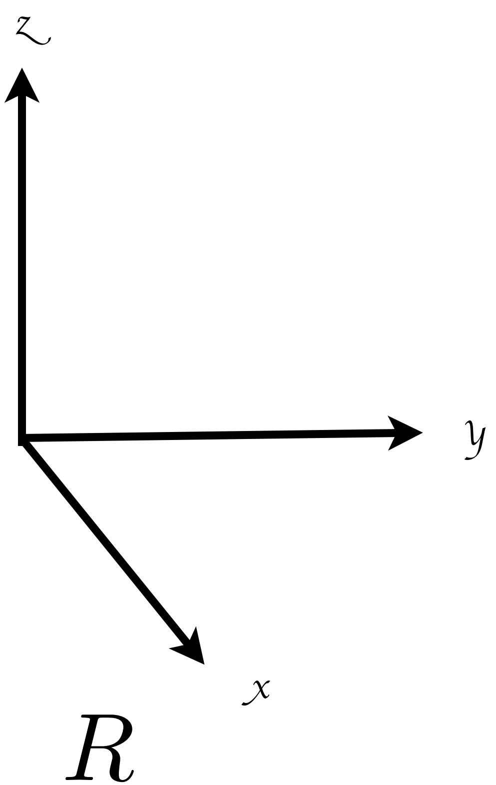
Hyperbolic Geometry and Special Relativity





Classical Mechanics

- To solve physical problems we must choose a reference frame



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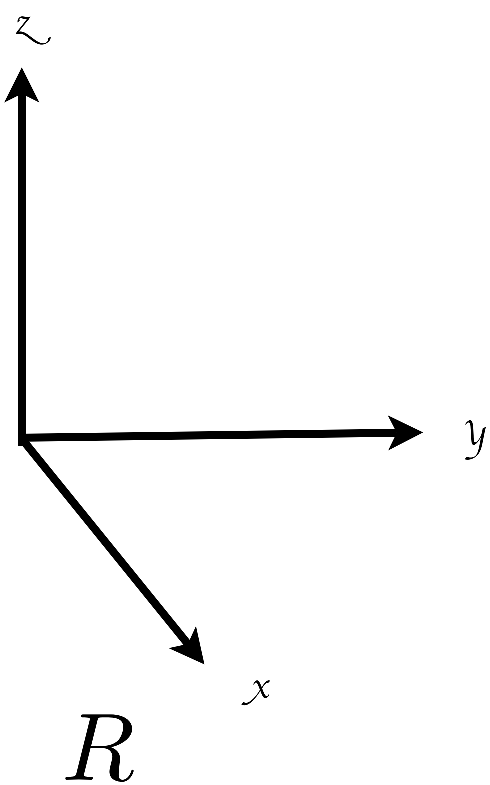

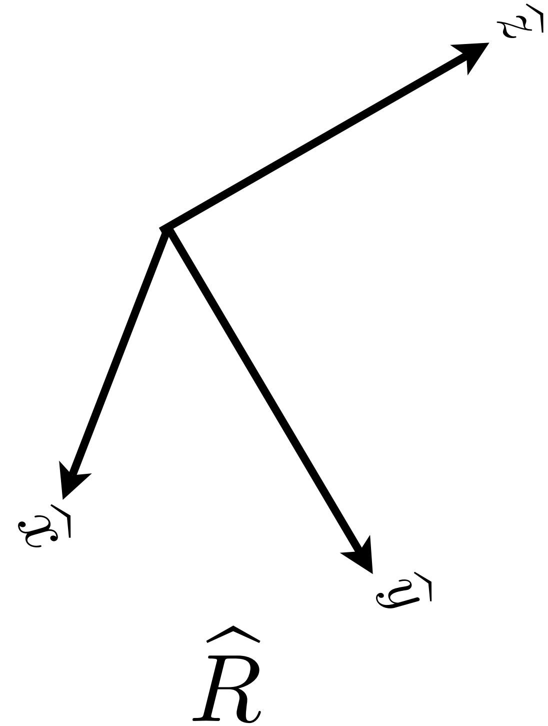

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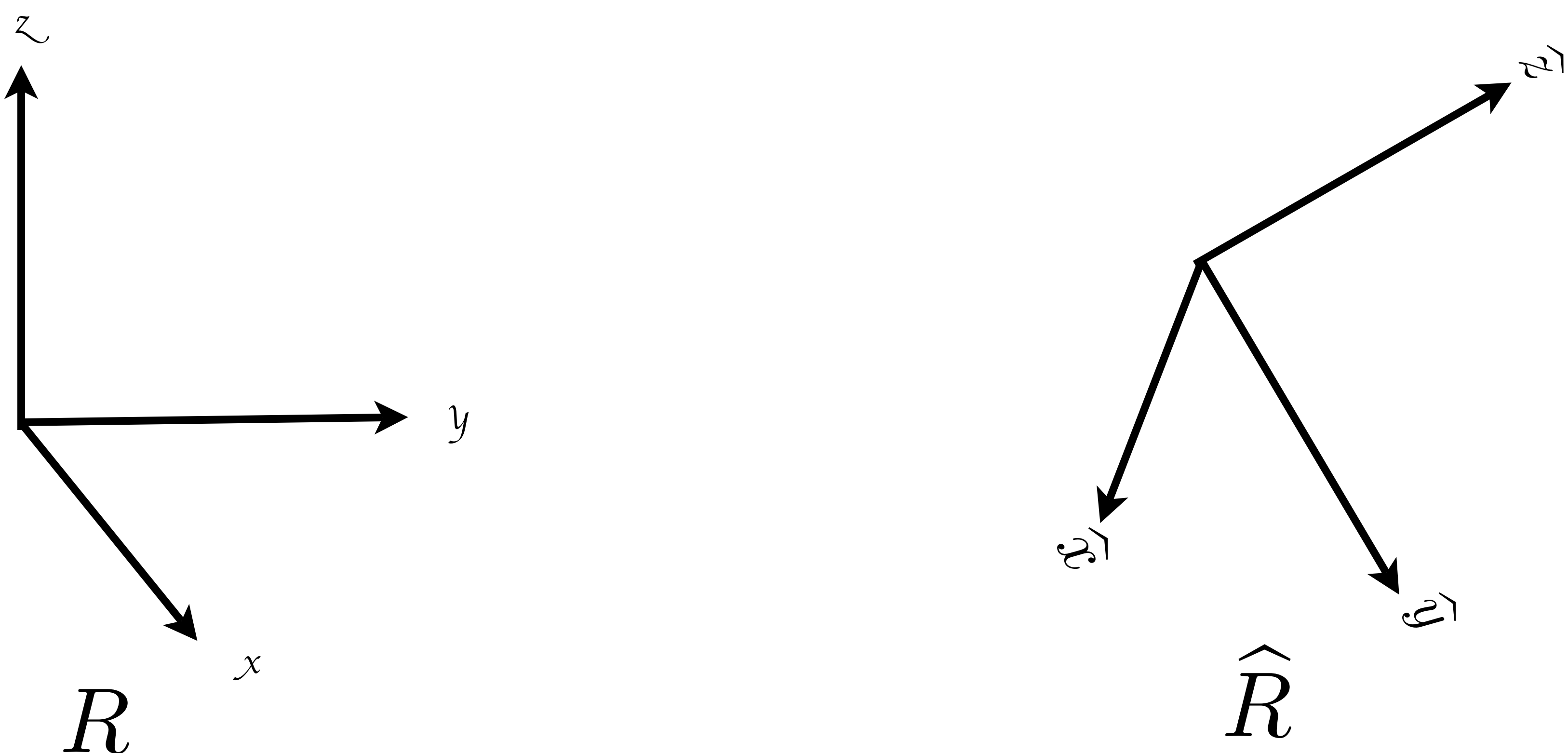

$$\mathbf{x} = H\widehat{\mathbf{x}} + T$$

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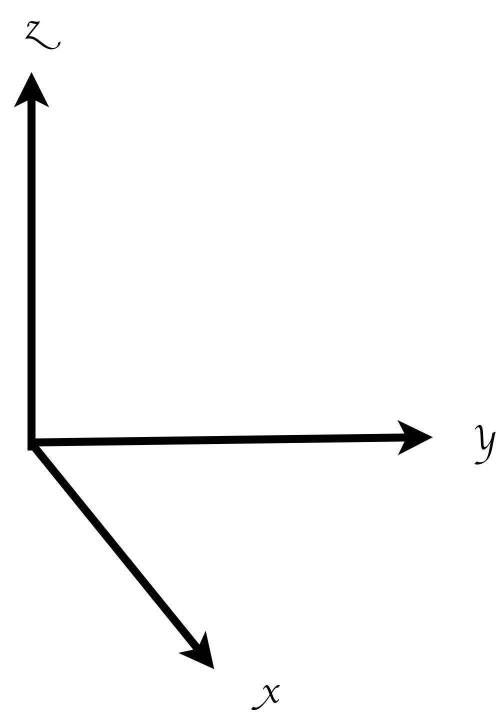

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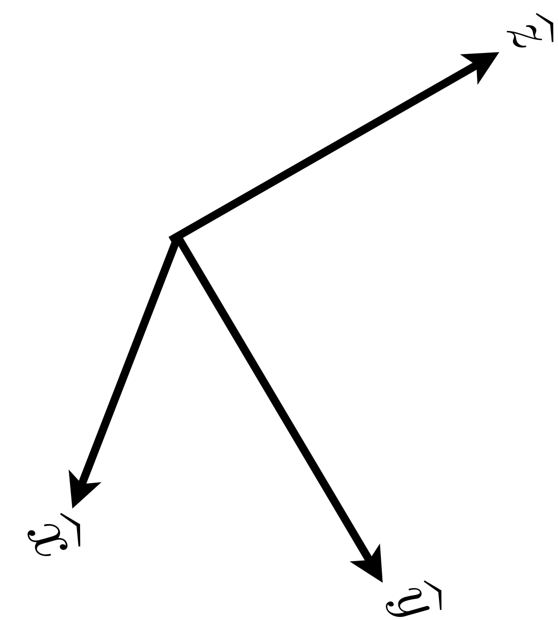


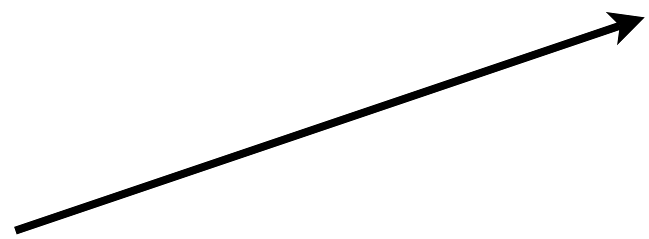
$\widehat{R}$

$$\mathbf{x} = H\widehat{\mathbf{x}} + T$$

$$H \in SO(n) \qquad T \in \mathbb{R}^n$$

Classical Mechanics

- Newton's 1st Law: w/o forces, bodies move in straight lines w/o accel.



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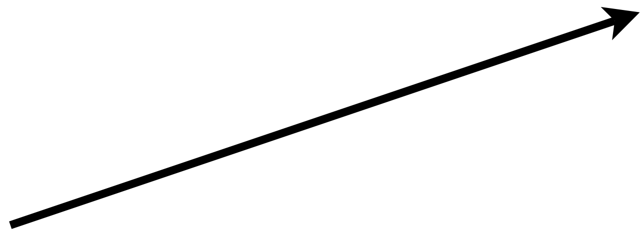

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inertial frames

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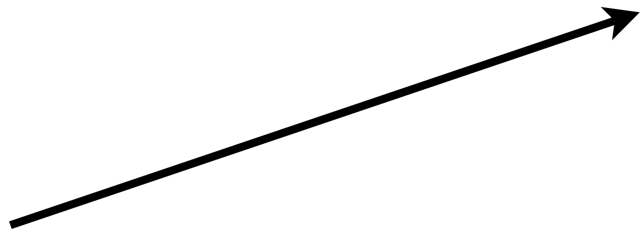

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inertial frames

- *Galilean Transformations* are affine change of coords. that preserve Newton's laws.

Classical Mechanics

•*Galilean Transformations* are affine change of coords. that preserve Newton's laws.

$$\begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ v_1 & & & \\ v_2 & & H & \\ v_3 & & & \end{pmatrix} \begin{pmatrix} \widehat{t} \\ \widehat{x} \\ \widehat{y} \\ \widehat{z} \end{pmatrix} + C$$

$v\ H\ C$ are constant

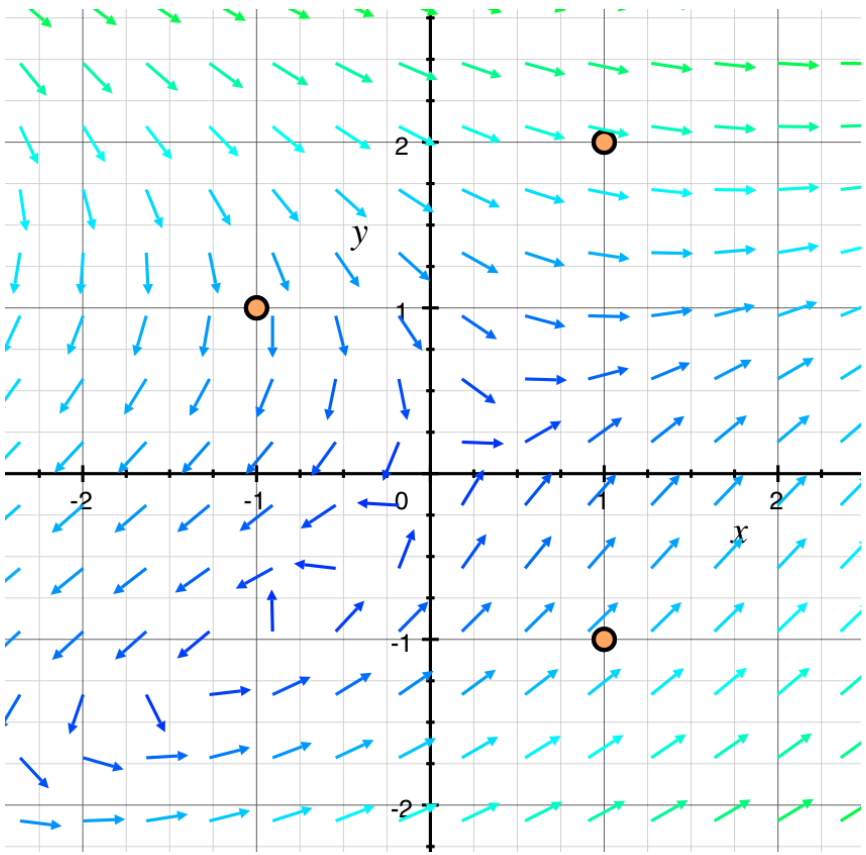
$$H \in SO(n)$$

•*Galilean Transformations* preserve time intervals and (euclidean) distance in space

Maxwell's Equations

E: electric field

B: magnetic field



1. Superposition
2. A stationary point generates no magnetic field and

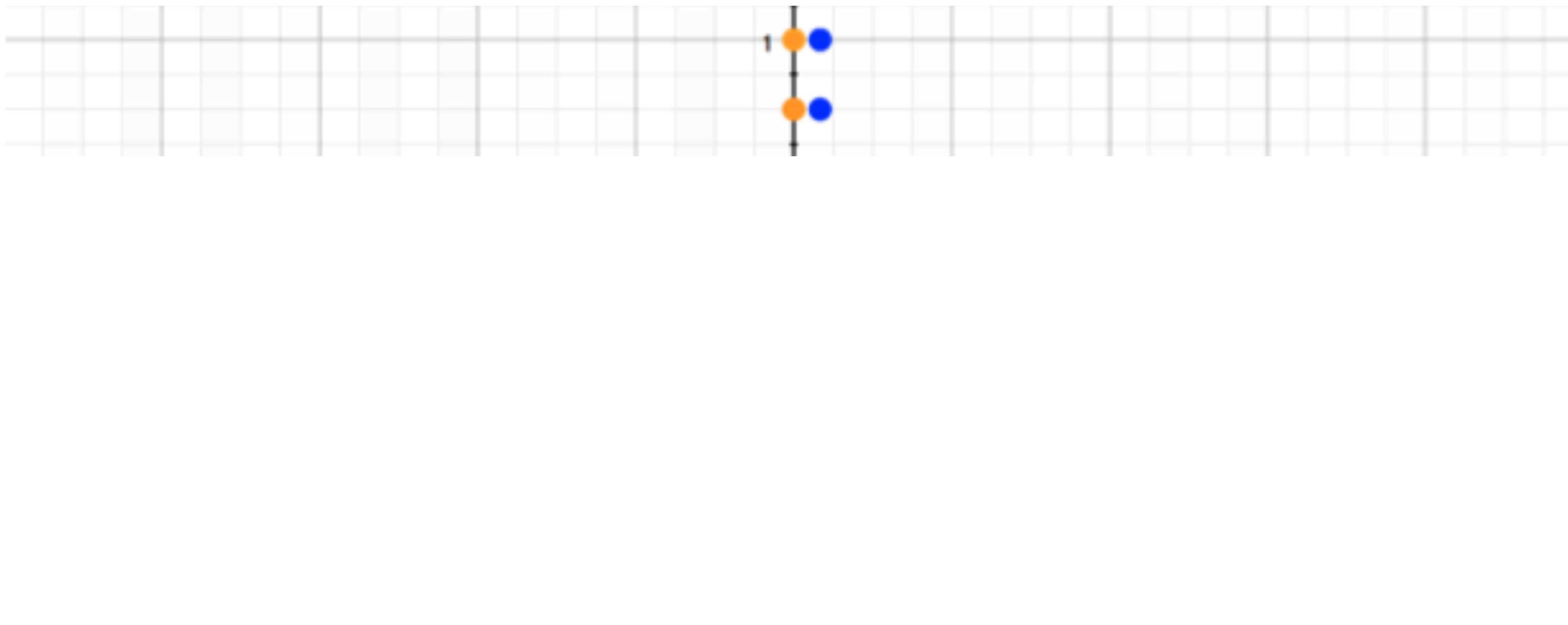
$$\mathbf{E} = \frac{ke\mathbf{r}}{r^3}$$

3. A point charge with velocity v generates a magnetic field

$$\mathbf{B} = \frac{k'e(\mathbf{v} \times \mathbf{r})}{r^3}$$

Corollary (Lorentz): $\mathbf{F} = e(\mathbf{E} + \mathbf{v} \times \mathbf{B})$

Maxwell's Equations are not invariant



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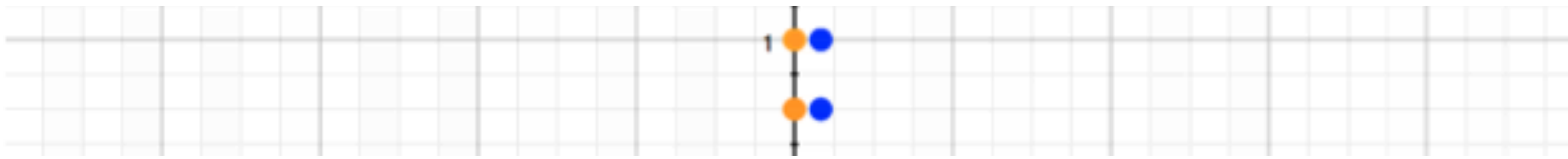

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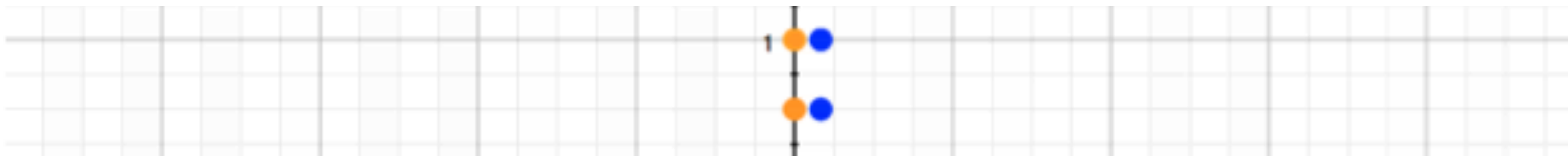

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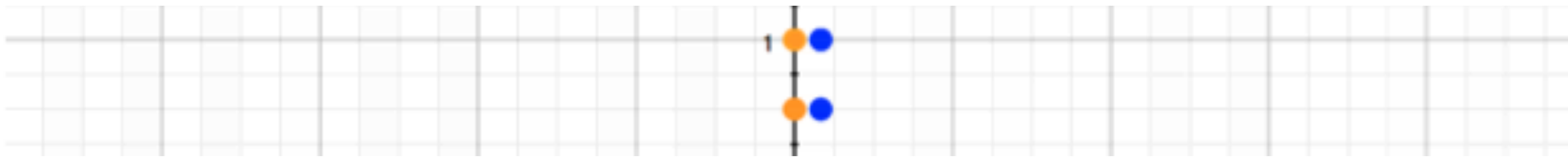

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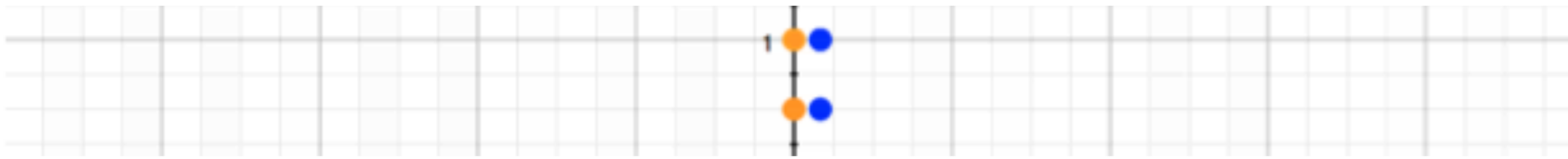

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and by switching the roles: $$\mathbf{E} = \widehat{\mathbf{E}} - \mathbf{v} \times \widehat{\mathbf{B}}$$

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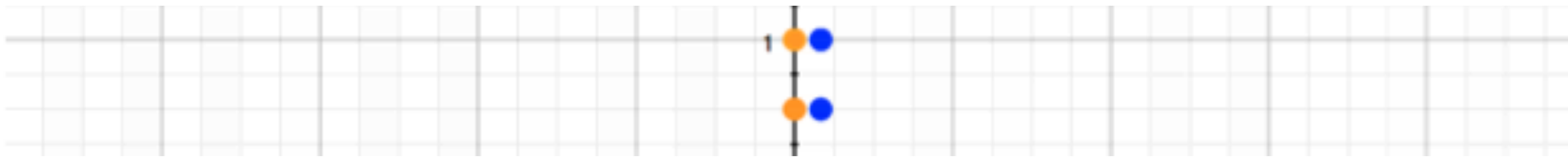

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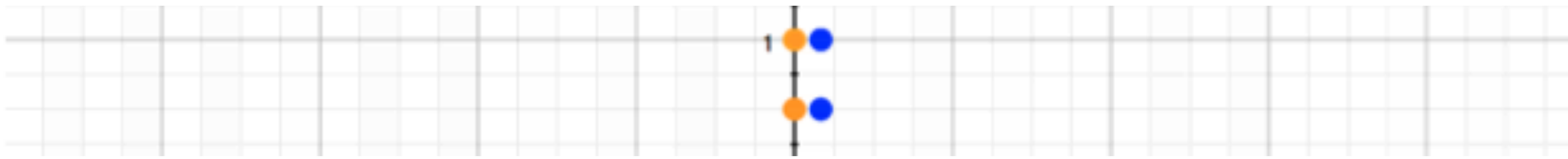

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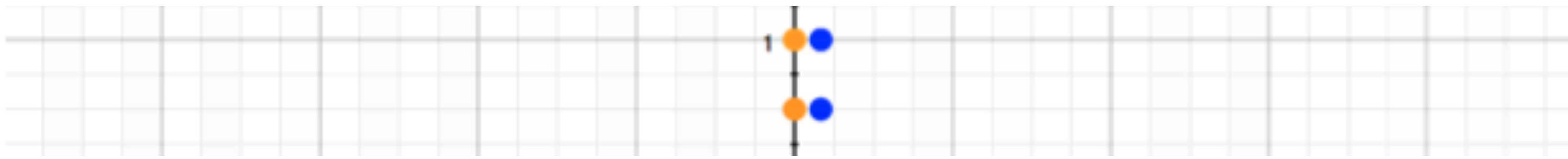

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but: $\mathbf{B} = 0$ and $\widehat{\mathbf{B}} = -\dfrac{\mu_0 e(\mathbf{v} \times \mathbf{r})}{4\pi r^3}$ by (EM 3).

Lorentz Transformations

We assume:

- Transformation is affine (Newton's 1st Law is preserved)
- Light travels at constant velocity and in straight lines. Hence photon world lines are of the form:

- No physical effect is transferred faster than light.
- Only (not accelerative) relative motion can be detected.

Lorentz Transformations

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- Transformation is affine (Newton's 1st Law is preserved)
- Light travels at constant velocity and in straight lines. Hence photon world lines are of the form:

$$\mathbf{x} = \mathbf{u}t + \mathbf{a}$$

$$||\mathbf{u}||^2 = c^2$$

- No physical effect is transferred faster than light.
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Lorentz Transformations are affine

$$\mathbf{x} = L\widehat{\mathbf{x}} + C$$

$$L = \begin{pmatrix} \gamma & - & - & - \\ - & - & - & - \\ - & - & - & - \\ - & - & - & - \end{pmatrix}$$

$$t = \gamma\widehat{t} + \text{constant}$$

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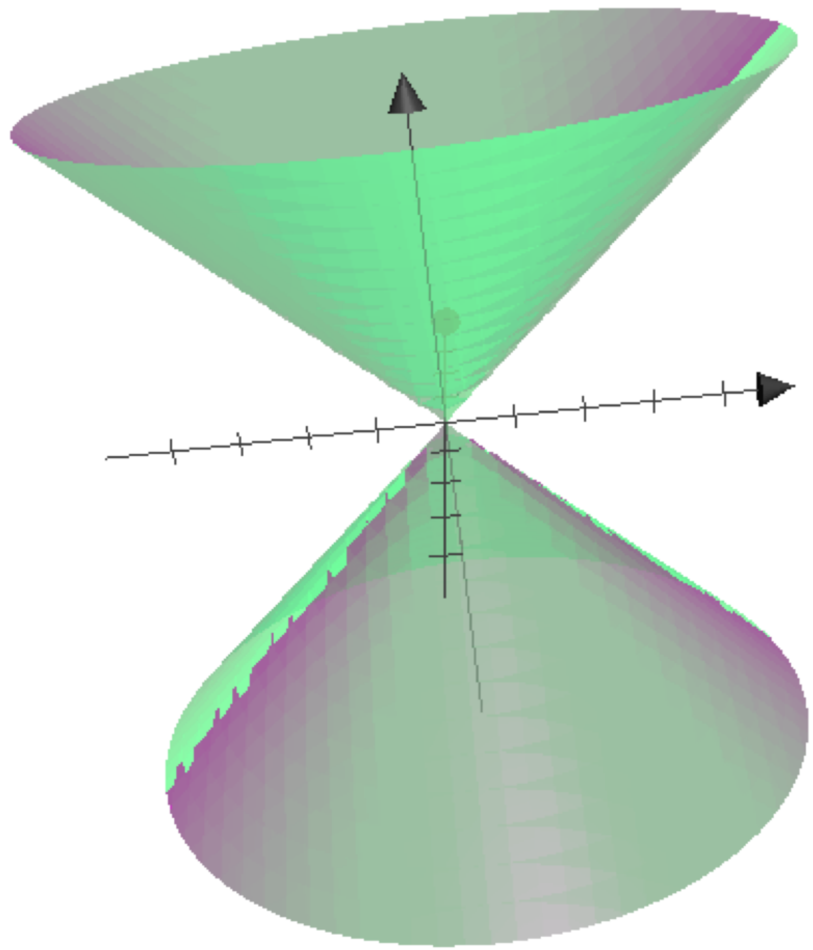
time dilation

$$L = \begin{pmatrix} \gamma & - & - & - \\ - & - & - & - \\ - & - & - & - \\ - & - & - & - \end{pmatrix}$$

$$t = \gamma\widehat{t} + \text{constant}$$

Lorentz Transformations have conical photon world lines

$$ct^2 - x^2 - y^2 - z^2 = 0$$



What's invariant?

The world lines of photons

Event E_i has $\mathbf{O}$ coordinates $\mathbf{x}_i$.

Event E_i has $\widehat{\mathbf{O}}$ coordinates $\widehat{\mathbf{x}}_i$.

They are on the world line of a photon iff:

$$D = cT$$

spatial displacement → D ; T ← *temporal displacement*

$$\begin{aligned} c^2(t_2 - t_1)^2 - (x_2 - x_1)^2 - (y_2 - y_1)^2 - (z_2 - z_1)^2 &= 0 \\ c^2(\widehat{t}_2 - \widehat{t}_1)^2 - (\widehat{x}_2 - \widehat{x}_1)^2 - \ldots &= 0 \end{aligned}$$

World Lines of Photons

$$X = \begin{pmatrix} ct_2 \\ x_2 \\ y_2 \\ z_2 \end{pmatrix} - \begin{pmatrix} ct_1 \\ x_1 \\ y_1 \\ z_1 \end{pmatrix} \qquad \widehat{X} = \begin{pmatrix} \widehat{ct_2} \\ \widehat{x_2} \\ \widehat{y_2} \\ \widehat{z_2} \end{pmatrix} - \begin{pmatrix} \widehat{ct_1} \\ \widehat{x_1} \\ \widehat{y_1} \\ \widehat{z_1} \end{pmatrix}$$

$$g = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Need:

$$X^t g X = 0 \Leftrightarrow \widehat{X}^T g \widehat{X} = 0$$

If $\widehat{X} = LX$ then:

$$X^t g X = 0 \Leftrightarrow X^T L^T g L X = 0$$

Minkowski Space $\mathbb{R}^{1,3}$

$$x \circ y = -x_1 y_1 + x_2 y_2 + x_3 y_3 + x_4 y_4$$

$$||x|| = \sqrt{x \circ x}$$

$$x \circ y = ||x|| \, ||y|| \cosh \eta$$

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$$||x|| > 0$$ spacelike

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$\|\|x\|\| > 0$	spacelike
$\|\|x\|\|$ imaginary	timelike

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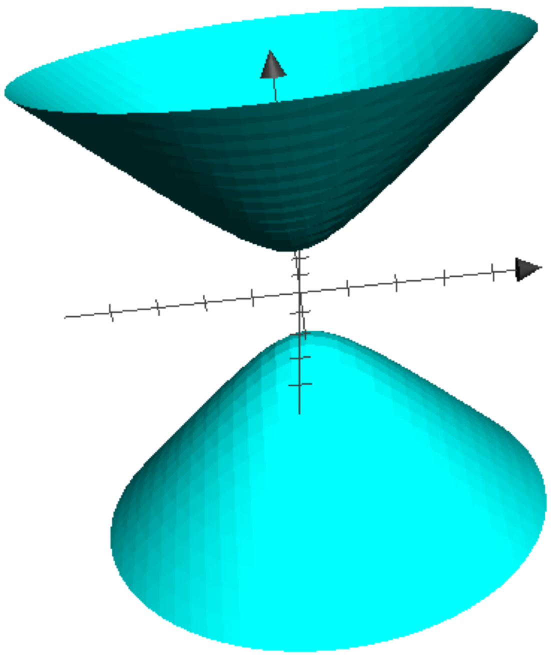
$\|\|x\|\| > 0$	spacelike
$\|\|x\|\|$ imaginary	timelike
$\phi(x) \circ \phi(y) = x \circ y$	Lorentz transformation

Minkowski Space $\mathbb{R}^{1,3}$

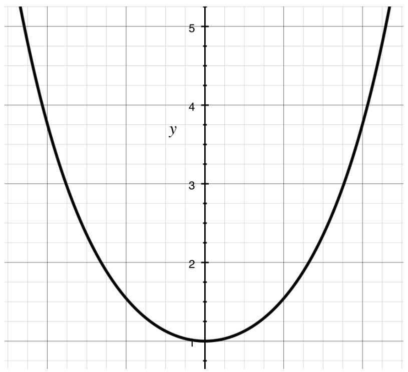
$$F^3 = \{x \in \mathbb{R}^{1,3} : ||x||^2 = -1\}$$



$$x \circ y = ||x|| ||y|| \cosh \eta$$

$$d_H(x, y) = \eta(x, y)$$

time-like angle



Minkowski Space

<u>Theorem:</u> F^3 with d_H is isometric to $\mathbb{H}^3$