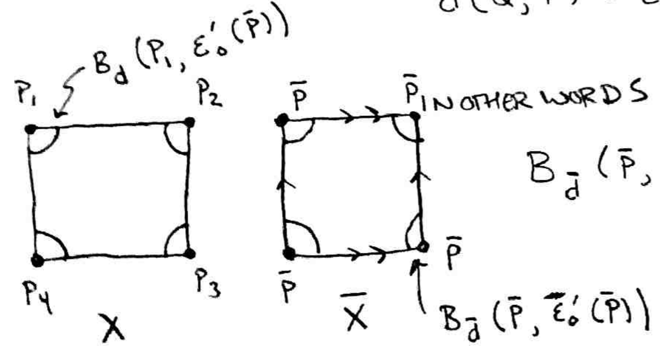
GOAL: Let X be a polygon in $\mathbb{E}^2$ and let $\bar{X}$ be obtained by gluing edges in pairs via isometries. Let $P \in X$ we wish to show that $\exists\, \varepsilon > 0$ s.t. $B_{\bar{d}}(\bar{P}, \varepsilon)$ is isometric to the result of gluing disc sectors in $\mathbb{E}^2$ together. (If the angles in X around $P' \in \bar{P}$ sum to 2π then the result is a euclidean disc)

METHOD We make extensive use of Lemma 4.5 from Bonahon: $\exists\, \varepsilon_0(P)$ s.t. $\forall\, \varepsilon < \varepsilon_0(P)$ and $\forall\, Q \in X$ $\bar{d}(\bar{P}, \bar{Q}) < \varepsilon \iff \exists\, P' \in \bar{P}$ with $d(P', Q) < \varepsilon$. 4.5

Let $\varepsilon_0'(\bar{P}) = \min_{P' \in \bar{P}} \varepsilon_0(P')$ and let $\varepsilon_0 = \frac{1}{3}\varepsilon_0'(\bar{P})$

OBSERVE : ⓞ $\forall\, Q \in X$ $\exists\, P' \in \bar{P}$ s.t.

$\bar{d}(\bar{Q}, \bar{P}) < \varepsilon_0'(\bar{P}) \iff \exists\, P' \in \bar{P}$ with $d(P', Q) < \varepsilon_0'(\bar{P})$



IN OTHER WORDS

$$B_{\bar{d}}(\bar{P}, \varepsilon_0'(\bar{P})) = \bigcup_{P' \in \bar{P}} B_d(P', \varepsilon_0'(\bar{P}))$$

1 IF $\bar{Q}, \bar{R} \in B_{\bar{d}}(\bar{P}, \varepsilon)$ (with $\varepsilon \leq \varepsilon_0'(\bar{P})$)

THEN $\boxed{\bar{d}(\bar{Q},\bar{R}) < 2\varepsilon}$

proof $\bar{d}(\bar{Q},\bar{R}) \leq \bar{d}(\bar{Q},\bar{P}) + \bar{d}(\bar{P},\bar{R}) < 2\varepsilon$ □

2 IF $Q \in B_d(P,\varepsilon)$ $R \in B_d(P',\varepsilon)$ with $P \neq P'$

and with $\varepsilon \leq \frac{1}{3}\varepsilon_0'(\bar{P})$ and $P, P' \in \bar{P}$ then

$\boxed{d(Q,R) \geq \frac{1}{3}\varepsilon_0'(\bar{P}).}$

proof: Since $P \neq P'$ they are on different edges

so $\varepsilon_0'(\bar{P}) \leq d(P,P') \leq d(P,Q) + d(Q,R) + d(R,P')$

$\Rightarrow$ $\varepsilon_0'(\bar{P}) < 2\varepsilon + d(Q,R) < \frac{2}{3}\varepsilon_0'(\bar{P}) + d(Q,R)$

$\Rightarrow$ $\frac{1}{3}\varepsilon_0'(\bar{P}) < d(Q,R)$

3 Suppose $Q, R \in B_d(P, \frac{1}{3}\varepsilon_0)$ We will show that $d(Q,R) \leq \bar{d}(\bar{Q},\bar{R})$ by beginning with a chain in $\bar{X}$ from $\bar{Q}$ to $\bar{R}$ and showing that we can shorten it to a chain in X lying entirely in $B_d(P,\varepsilon_0)$

<CONTINUED>

3 CONTINUED

Let $\bar{C} : \bar{Q} = \bar{x}_0, \bar{x}_1, \dots, \bar{x}_n = \bar{R}$ be a chain from $\bar{Q}$ to $\bar{R}$ in $\bar{X}$. By 1 we may assume $\ell(\bar{C}) < \frac{2}{3}\varepsilon_0 \leq \frac{2}{9}\varepsilon_0'(\bar{P})$.

a Each point $\bar{x}_i \in B_{\bar{d}}(\bar{P}, \varepsilon_0)$

<u>proof</u> By the triangle inequality

$$\bar{d}(\bar{P}, \bar{x}_i) \leq \bar{d}(\bar{P}, \bar{Q}) + \bar{d}(\bar{Q}, \bar{x}_1) + \dots + \bar{d}(\bar{x}_{i-1}, \bar{x}_i)$$
$$\leq \bar{d}(\bar{P}, \bar{Q}) + \ell(\bar{C})$$
$$< \frac{1}{3}\varepsilon_0 + \frac{2}{3}\varepsilon_0$$
$$= \varepsilon_0$$
$$\leq \frac{1}{3}\varepsilon_0'(\bar{P}) \quad \square$$

b We now move to X: let

$$C: \; Q' = x_0, \underset{\text{jump}}{} y_1, \underset{\text{teleport}}{} x_1, y_2, x_2, \dots, y_n = R'$$

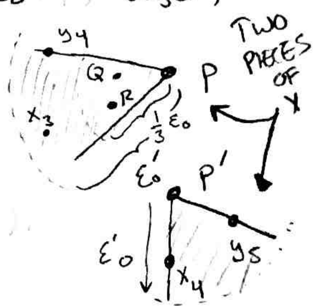
be a chain in X s.t. $\bar{Q}' = \bar{Q}$, $\bar{R}' = \bar{R}$, $\bar{y}_i = \bar{x}_i$

and $\ell(\bar{C}) = \sum d(x_{i-1}, y_i)$

Since $d(\bar{P}, \bar{x}_i) = d(\bar{P}, \bar{y}_i) < \varepsilon_0$ (by a)

by 4.5 $\exists\, P_i, P_i' \in \bar{P}$ with

$$d(P_i, x_i) < \varepsilon_0$$
$$d(P_i', y_i) < \varepsilon_0'$$



Hence, each point in C is "near" a point in $\bar{P}$

C Claim: $P_i = P'_{i+1}$

near x_i near y_i

jump in C

<u>proof</u> Suppose $P_i \neq P'_{i+1}$. By 2

$$d(x_i, y_i) \geq \frac{1}{3} \varepsilon'_0(\bar{P}) \quad \text{since } \varepsilon_0 = \frac{1}{3}\varepsilon'_0(\bar{P}).$$

Hence

$$\frac{1}{3} \varepsilon'_0(\bar{P}) \leq d(x_i, y_i) \leq \ell(\bar{C}) \leq \frac{2}{9} \varepsilon'_0(\bar{P})$$

$$\Rightarrow \quad \frac{1}{3} \leq \frac{2}{9} \quad \otimes$$

D We now set about creating a chain C' in X from $Q \in B_d(P, \frac{1}{3}\varepsilon_0)$ to $R \in B_d(P, \frac{1}{3}\varepsilon_0)$ which gives rise to a chain $\bar{C}'$ from $\bar{Q}$ to $\bar{R}$ of length at most $\ell(\bar{C})$ and which consists entirely of points interior to $B_d(P, \varepsilon_0)$ unless P is a vertex, in which case something more complicated can happen.