MA 314 Homework 4: Careful Reflection

1. Mortar

(1) A rotation of $\mathbb{E}^2$ about a point x_0 is a function $R\colon \mathbb{E}^2 \to \mathbb{E}^2$ such that the map T defined by $T(x) = R(x) - x_0$ is a rotation around the origin (i.e. is a linear map whose matrix with respect to the standard basis). Prove that a rotation about x_0 preserves each of the sets $\{x : ||x - x_0|| = r\}$ for fixed r.

(2) Give an example of an isometry $T : \mathbb{E}^2 \to \mathbb{E}^2$ such that the fixed point set $\{x : T(x) = x\}$ is empty but for which there is a line L such that $T(L) = L$ (i.e. T fixes a line).

(3) Give an example of an isometry $\mathbb{E}^3 \to \mathbb{E}^3$ with empty fixed point set but which fixes a line.

2. Bricks

(1) Recall that if $x \in \mathbb{E}^2$ and if $L \subset \mathbb{E}^2$ is a line, then the reflection $R_L(x)$ of x across the line L is x if $x \in L$ and otherwise is the point on the opposite side of L from x and the same distance from L as x is. Give a completely rigorous proof that R_L is an isometry of $\mathbb{E}^2$.

(2) Prove that every isometry $\mathbb{E}^2 \to \mathbb{E}^2$ is the composition of at most 3 reflections across lines.

(3) In the previous problem, you proved that every isometry can be written as the composition of reflections across lines. Of course, there may be many ways of doing this, using varying numbers of reflections. Prove, however, that if you have two ways of expressing an isometry as the composition of reflections than the number of reflections is either odd for both or even for both.

(4) Outline a proof that every isometry $\mathbb{E}^3 \to \mathbb{E}^3$ is the composition of translations, rotations, and reflections (across planes).