MA 314 Homework 3: A Length-y one.

In class we defined infimum and supremum of subsets of $\mathbb{R}$. The first few problems are intended to reinforce those ideas. The other problems on this homework will get you ready for discussing continuous functions.

1. Reading

- Schwartz: Sections 1.1 - 1.5
- Bonahon: Section 1.4

The reading from Schwartz's book gives you an overview of some of what we'll be doing. The reading from Bonahon's book introduces euclidean geometry in the way we'll need it for the rest of the course.

2. Mortar

(1) Do Exercise 1.1 on page 7 of Bonahon. (This shows that the metric distance between points in $\mathbb{R}^2$ is achieved by a path. You may assume, without proof, that a rotation is an isometry of $\mathbb{E}^2$ and that rotations preserve the property of being a straight line.)

(2) Do Exercise 1.2 on page 7 of Bonahon. (This shows that a straight line minimizes euclidean distance between points.)

(Even though we outlined how to do these problems in class, I'd like you to think through them again. We'll be doing something similar in other geometries, so it's important to have a good grasp on these problems now.)

3. Bricks

(1) Give a rigorous proof of the fact that the path metric on a path-connected subset $U \subset \mathbb{R}^n$ is a metric. (Recall that "path" means "piece-wise C^1 path".) (This is essentially Exercise 1.3 on page 7 of Bonahon. You'll want to reference the two mortar problems from this assignment.)

(2) A **translation** of $\mathbb{E}^n$ (i.e. $\mathbb{R}^n$ with the euclidean path metric) is a function $T\colon \mathbb{R}^n \to \mathbb{R}^n$ of the form $T(x) = x + a$ for some constant $a \in \mathbb{R}^n$. A **line** is the image of a path with constant, but non-zero,

derivative. Prove that translations are isometries of $\mathbb{E}^n$ and that they preserve lines.

(3) We need to generalize the notion of path distance to spaces where we may not be able to define the notion of "derivative". We'll call the resulting notion "grasshopper distance". Let X be a non-empty set.

A **chain** or **discrete path** in X is a finite sequence $x_0, x_1, \dots, x_n$ of points in X. Suppose that $\delta\colon X \times X \to [0,\infty) \subset \mathbb{R}$ is a symmetric real-valued function. That is, $\delta(x,y) = \delta(y,x)$ for all $x, y \in X$. Think of δ as being "almost a distance function". We want to use it to define something like a path metric. Define the **δ-length** $\delta(C)$ of C to be $\sum_{i=1}^{n} \delta(x_{i-1}, x_i)$. The **grasshopper distance** from $x \in X$ to $y \in X$ is

$$d(x,y) = \inf_C \delta(C)$$

where the infimum is taken over all chains C in X which begin at x and end at y.

(a) Prove that d satisfies the symmetry property and the triangle inequality.

(b) Give an example of δ for $X = \mathbb{R}$. Give an informal description of the distances between points as they move along $\mathbb{R}$.

(This is an exercise we'll make good use of later in the course.)

(4) (For those who have seen metric spaces before) Do Exercise 1.11 on page 9 of Bonahon.