MA 314 Homework 2: What Sup?

Metric spaces will be the foundation of all of our geometric work this semester. They are intended to axiomatize the idea of "space with a distance".

1. Reading

- Bonahon: Appendix T.2

2. Mortar

These problems are intended to give you some practice with basic concepts. They will often involve calculation, rarely involve new ideas, and won't be graded. However, your answers will be collected!

(1) Find the sup and inf of the following subsets of $\mathbb{R}$. Also, say whether or not the sup and inf are elements of the set or not.

   (a) $(0,1)$ (this is the interval consisting of precisely those $x \in \mathbb{R}$ such that $0 < x < 1$)

   (b) $[0,1)$ (this is the interval consisting of precisely those $x \in \mathbb{R}$ such that $0 \leq x < 1$)

   (c) $(0,1] \cup [2,3)$

   (d) $\{7\}$

   (e) $\{\frac{1}{n} : n \in \mathbb{N}\}$. (Recall that $\mathbb{N} = \{1,2,3,\ldots\}$ is the set of natural numbers.)

   (f) $\{1 - \frac{1}{n} : n \in \mathbb{N}\}$.

(2) Give an example of two sequences (x_n) and (y_n) of real numbers such that for all n, $x_n \neq y_n$, but

$$\inf_{n \in \mathbb{N}} \{x_n - y_n\} = 0.$$

3. Bricks

These problems are intended to require more thought and less calculation.

(1) Let (X,d) be a metric space. A sequence (x_n) in X has the **limit** $x \in X$ if for all $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $d(x_n, x) < \varepsilon$. We also say that (x_n) **converges** to x.

(a) Let (x_n) be a constant sequence. That is, there exists $x \in X$ such that for all n, $x_n = x$. Prove that (x_n) converges to x.

(b) Suppose that (x_n) converges to $x \in X$. Prove that the limit is unique. That is, prove that if (x_n) also converges to $y \in X$, then $x = y$. (Hint: Do a proof by contradiction.)

(c) (challenging!) Let (Y, d_Y) also be a metric space. Define a function $f\colon X \to Y$ to be **sequentially continuous** if whenever (x_n) is a sequence in X converging to $x \in X$, the sequence $(f(x_n))$ in Y converges to $f(x)$.

Prove that a function $f\colon X \to Y$ is continuous if and only if it is sequentially continuous. (For the definition of continuous function use Definition 2.3 on page 24 of Schwartz) or, equivalently, on page 4 of Bonahon.)

(2) Find an example of an open subset $U \subset \mathbb{R}^2$ such that there are two points $a, b \in U$ such that no path in U achieves the path distance (in U) from a to b.

(Examples like these show that in the definition of path distance we really do need to write infimum and not just "minimum".)

(3) Show that every two open intervals in $\mathbb{R}$ are homeomorphic. (Remember to consider both intervals of finite length and of infinite length.)

(Hint: First show that every interval of the form (a,b) is homeomorphic to the interval $(0,1)$. Then show that the interval $(-\pi/2, \pi/2)$ is homeomorphic to the interval $(-\infty, \infty)$. Finally, deal with intervals of the form $(-\infty, a)$ and (b, ∞). As you look for homeomorphisms, you don't need to get too crazy – mostly linear functions and the occasional trig function will do.)

(4) (For those who have seen metric spaces before) If (X, d_X) and (Y, d_Y) are compact metric spaces, let $\mathscr{F}(X,Y)$ denote the set of all continuous functions $X \to Y$. We can give $\mathscr{F}(X,Y)$ a metric by saying $d(f,g) = \sup_{x \in X} d_Y(f(x), g(x))$ where $f, g\colon X \to Y$. Explain what would it would mean for there to be a path from a function $f \in \mathscr{F}(X,Y)$ to a function $g \in \mathscr{F}(X,Y)$. Why might one want to be able to assign lengths to such paths?