MA 314 Homework 19: Acting Freely

1. Mortar

(1) Read Chapter 7 of Bonahon and summarize the proof of the fact that if a group action has a fundamental domain, then it is discontinuous.

(2) Sketch a picture of a locally finite polygon in $\mathbb{E}^2$ with infinitely many edges which tiles $\mathbb{E}^2$. (Hint: consider a polygonal approximation to the region bounded by $y = x^3$ and $y = x^3 + 1$.)

2. Bricks

Given how specialized our description of group actions has been, you may wonder how prevalent discontinuous free actions are. The purpose of these problems is to prove:

Theorem. Let G be a group. Then there is a path metric space (X, d) such that G acts on X discontinuously, freely, and by isometries.

(1) Let G be a group. A set $S \subset G$ is a **generating set** for G if every element of G can be written as the composition of finitely many elements of S and inverses of elements of S. Prove that every group G has a generating set. (There is a silly way of doing this.)

(2) Let Γ be a graph which has a vertex v_g for each element $g \in G$. Two vertices v_a and v_b are joined by an edge, if there is an element $s \in S$ (for a generating set S) such that $a = bs$. Prove that the graph Γ is connected (i.e. that there is a sequence of edges joining any two vertices of Γ)

(3) Give each edge of Γ a length of 1 and view it as a copy of the unit interval $[0,1] \subset \mathbb{E}^1$. Define the distance between two vertices of Γ to be the minimal number of edges in an edge path joining them. Prove that Γ is path metric space.

(4) Define an action of G on Γ by $gv_a = v_{ga}$ where $g \in G$ and v_a is a vertex of Γ. Prove that this is a group action. (i.e. that

- If $g, g' \in G$, then $(gg')v_a = g(g'v_a)$
- If $1 \in g$ is the identity element, then $1v_a = v_a$ for all vertices $v_a \in \Gamma$.

As part of your answer, show how to extend the definition of the action from the vertices of Γ to all points in Γ.

(5) Prove that the action of G on Γ is free, discontinuous, and by isometries. (Hint: observe that the minimum distance from any vertex to any other vertex of Γ is 1.)

(6) The **free group** F_2 on two generators is the group consisting of all "words" in two symbols (say a and b) and their inverse symbols A and B. For example, $aaaBBa$ is an example of a word in F_2. Of course $aA = Aa = 1$ and $Bb = bB = 1$. The group F_2 is generated by a and b. Sketch a portion of the graph Γ corresponding to $S = \{a, b\}$.