MA 314 Homework 17: Building spaces by trading places

These problems are intended to help improve your intuition for gluing polygons in euclidean, spherical and hyperbolic space.

1. MORTAR

(1) Summarize the major steps needed to prove that any surface created by gluing edges of polygons in pairs by isometries is locally isometric to $\mathbb{E}^2$, $\mathbb{H}^2$, or S^2 if the angle sum around each vertex is 2π.

(2) Give an example of polygons in $\mathbb{E}^2$ which glue up (by edge isometries) to be a sphere. What cone angles can you achieve?

2. BRICKS

(1) Suppose that $X \subset \mathbb{E}^2$ is a polygon without vertices (remember that the edges of X have to be complete geodesics). Let $\overline{X}$ be obtained by gluing the edges of X pairwise by isometries. What (topological) surfaces can you get?

(2) In the upper half-plane model of $\mathbb{H}^2 \subset \mathbb{C}$, let X be the region bounded by the geodesics $E_1 : x = -1$ and $E_2 : x = +1$. Let $\phi : E_1 \to E_2$ be the map $\phi(z) = z + 2$. Note that ϕ is an isometry since it is a homothety. Let $\overline{X}$ be the glued up surface.

   (a) Explain why $\overline{X}$ is a cylinder (i.e. a bi-infinite annulus)

   (b) Let $P_1(y)$ and $P_2(y)$ be the points of E_1 and E_2 at height y. Observe that in $\overline{X}$, the points $\overline{P}_1(y)$ and $\overline{P}_2(y)$ are the same.

      (i) What is the distance in X between $P_1(y)$ and $P_2(y)$? (You may want to go back to chapter 2 of Bonahon to find an easy-to-use distance formula). Note that this distance is achieved by a hyperbolic geodesic $g(y)$ in X which glues up in $\overline{X}$ to be a geodesic loop $\overline{g}(y)$ of the same length.

      (ii) What happens to the length of the loop $\overline{g}(y)$ as y approaches 0? as y approaches ∞?

      (iii) For what value of y is the length of the loop $\overline{g}(y)$ the smallest? the largest?

(iv) Use your answers to the previous questions to sketch a picture of the glued up cylinder $\overline{X}$ which captures something of its geometry.

(3) In the upper half-plane model of $\mathbb{H}^2 \subset \mathbb{C}$, let X be the region bounded by the geodesics $E_1 : |z| = 1$ and $E_2 : |z| = 2$. Let $\phi \colon E_1 \to E_2$ be the map $\phi(z) = 2z$. Note that ϕ is an isometry since it is a homothety. Let $\overline{X}$ be the glued up surface.

(a) Explain why $\overline{X}$ is a cylinder (i.e. a bi-infinite annulus)

(b) Let $P_1(\theta)$ and $P_2(\theta)$ be the points of E_1 and E_2 an angle of θ from the positive x-axis. Observe that in $\overline{X}$, the points $\overline{P}_1(\theta)$ and $\overline{P}_2(\theta)$ are the same.

(i) What is the distance in X between $P_1(\theta)$ and $P_2(\theta)$? (You may want to go back to chapter 2 of Bonahon to find an easy-to-use distance formula). Note that this distance is achieved by a hyperbolic geodesic $g(\theta)$ in X which glues up in $\overline{X}$ to be a geodesic loop $\overline{g}(\theta)$ of the same length.

(ii) What happens to the length of the loop $\overline{g}(\theta)$ as θ approaches 0? as θ approaches π?

(iii) For what value of θ is the length of the loop $\overline{g}(\theta)$ the smallest?

(iv) Use your answers to the previous questions to sketch a picture of the glued up cylinder $\overline{X}$ which captures something of its geometry.