MA 314 Homework 16: Gluing Polygons

These problems are intended to help improve your intuition for gluing polygons in euclidean, spherical and hyperbolic space.

1. Mortar

(1) Draw a bounded (i.e. no edges of infinite length) polygon in $\mathbb{E}^2$, specify gluing maps to glue the edges together in pairs. Sketch a picture of the resulting object up to homeomorphism. Draw some ε balls centered at various points on the glued-up polygon. Some of the ε-balls you draw should intersect the edges of the polygon and wrap around.

(2) Draw an unbounded (i.e. at least one edge has infinite length) polygon in $\mathbb{E}^2$, specify gluing maps to glue the edges together in pairs. Sketch a picture of the resulting object up to homeomorphism. Draw some ε balls centered at various points on the glued-up polygon. Some of the ε-balls you draw should intersect the edges of the polygon and wrap around.

(3) Draw a, possibly disconnected, polygon on S^2 with edges consisting of portions of great circles and specify gluing maps so that the resulting surface is homeomorphic to the projective plane. (Hint: the projective plane can be obtained by gluing edges of a polygon with just two sides). What do the angles sum to around the vertices?

2. Bricks

(1) In class we proved Lemma 4.5 from Bonahon. Use this lemma to prove that (under our standard gluing assumptions) the grasshopper metric $\overline{d}$ is actually a metric, and not just a pseudo-metric.

(2) If $x \in X$ is a point in a metric space. A **local isometry** at x to a point y in a metric space Y is an isometry $f\colon B_\varepsilon(x) \to B_\varepsilon(y)$ (for some $\varepsilon > 0$) where $f(x) = y$.

Let $X \subset \mathbb{E}^2$ be a polygon and let $\overline{X}$ be the result of gluing the edges of X in pairs via isometries. Prove the following concerning the map $\phi\colon X \to \overline{X}$ given by $\phi(P) = \phi(\overline{P})$.

(a) ϕ is continuous. (You may use whatever definition of continuity from earlier in the semester you find most convenient)

(b) For all points x in the interior of X, the map ϕ is a local isometry at x.

(c) If $X = [-1,1] \times [-1,1] \subset \mathbb{E}^2$ and if $\overline{X}$ is the torus obtained by the usual gluing, then there is a local isometry from every point of $\overline{X}$ to a point of $\mathbb{E}^2$.

(d) Give an example of a polygon $X \subset \mathbb{E}^2$ and a gluing given by isometries on pairs of edges so that there is a point $\bar{x} \in \overline{X}$ where there is no local isometry at $\bar{x}$ to a point of $\mathbb{E}^2$. (See here for a hint.)

(3) At the end of class, we said that if $X \subset \mathbb{E}^2$ is non-convex, then we take d to be the path metric on X rather than the euclidean metric. Speculate as to why we would do this. What could go wrong if we don't?