MA 398 Homework 1: Metric Spaces go the distance!

Metric spaces will be the foundation of all of our geometric work this semester. They are intended to axiomatize the idea of "space with a distance".

1. Reading

- Bonahon: Preface and Sections 1.1 - 1.3
- Schwartz: Sections 2.1 - 2.4

Remark: For the time being you should pay attention mostly to the axioms for metric spaces. We'll worry about the other definitions later. Notice that Bonahon lists 4 axioms but Schwartz lists only 3 – how do you reconcile that?

What Bonahon calls a *semi-metric* we will sometimes call a *pseudo-metric*. Our own Ben Mathes has a particular fondness for things called "partial pseudo-metrics". We won't discuss those though...

2. Mortar

These problems are intended to give you some practice with basic concepts. They will often involve calculation, rarely involve new ideas, and won't be graded. However, your answers will be collected!

Show that the following are metric spaces:

(1) The real numbers $\mathbb{R}$ with metric d defined by $d(x,y) = |x-y|$.

(2) Any set X with a metric $d\colon X \times X \to \mathbb{R}$ defined by:

$$d(x,y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y. \end{cases}$$

This metric is called the *discrete metric*

(3) Suppose that (X,d) is a metric space and that $Y \subset X$. Define $d\big|_Y\colon Y \times Y \to \mathbb{R}$ by

$$d\big|_Y(x,y) = d(x,y)$$

Prove that $d\big|_Y$ is a metric. It is called the *restriction* of d to Y.

(4) Recall the definition of parameterized path in $\mathbb{R}^2$ from Vector Calculus: A parameterized path is a continuous function $\phi\colon [a,b] \to \mathbb{R}^2$. For example, the function $\phi(t) = (\cos t, \sin t)$ is a parameterization of the unit circle when $t \in [0, 2\pi]$. Find parameterized paths in $\mathbb{R}^2$ with the following images:

- the line segment between the points $(1,0)$ and $(3,5)$.

- the circle of radius 2 centered at the point $(1,1)$.
- the graph of a continuous function $f\colon [a,b] \to \mathbb{R}$. (For example, $f(x) = x^2$ for $x \in [-1,2]$.)

(Parameterized paths will play an extremely important role this semester. The purpose of this problem is to get you to remember (or possibly learn!) some basic examples.)

3. Bricks

These problems are intended to require more thought and less calculation.

(1) Give an example of a set X and a function $d\colon X \times X \to \mathbb{R}$ satisfying the following:

  (a) $d(x,y) = 0$ if and only if $x = y$.
  (b) For all $x,y,z \in X$,
  $$d(x,y) + d(y,z) \geq d(x,z).$$

  But such that there exist $x,y \in X$ such that $d(x,y) \neq d(y,x)$.

  Hint: You can do this with a space having exactly two points.

(2) Suppose that (X,d) is a metric space. Bonahon defines an **isometry** to be a bijection $f\colon X \to X$ such that for all $x,y \in X$, $d(x,y) = d(f(x),f(y))$. The purpose of this problem is to give an example showing that the requirement that f is a bijection does not follow from the distance-preserving property.

  Let $X \subset \mathbb{R}$ be the interval $[0,\infty)$ with the usual (euclidean) metric d. Find a function $f\colon X \to X$ which is not a bijection but which has the property that for all $x,y \in X$, $d(x,y) = d(f(x),f(y))$.

(3) (For students who have seen metric spaces before) Let (X,d) be a metric space. For non-empty subsets $A,B \subset X$ define $d_H(A,B)$ to be the infimal value of $\varepsilon \geq 0$ such that every point of A is distance at most ε from some point of B and every point of B is distance at most ε from some point of A. Prove that d_H is a pseudo-metric on the set of non-empty subsets of X. Can you find a (reasonably large) subset of $\mathscr{P}(X)$ (the power set of X) on which d_H is an actual metric?