MA 122: Weekly HW 10

Answer these questions on a separate sheet of paper. Remember that your work must be very neat and complete.

Problem A:

(1) Use the definition of the Riemann integral of one variable to calculate

$$\int_{-1}^{1} 3x\,dx.$$

Solution: There are several ways of doing this. Here is one.
By definition,

$$\int_{-1}^{1} 3x\,dx = \lim_{n\to\infty} f(x_i^*)\Delta x$$

where x_i^* is a sample point from the ith subinterval of $[-1,1]$ (which has been divided into n subintervals of equal length Δx). Choose x_i^* to be the right hand endpoint of the ith subinterval. Then

$$\begin{aligned} \Delta x &= (1-(-1))/n &&= 2/n \\ x_i^* &= -1+i\Delta x &&= -1+(2i)/n \end{aligned}$$

Consequently,

$$\begin{aligned} \textstyle\int_{-1}^{1} 3x\,dx &= \lim_{n\to\infty} \textstyle\sum_{i=1}^{n} 3(-1+i\Delta x)\Delta x \\ &= \lim_{n\to\infty} \textstyle\sum_{i=1}^{n} 3(-1+(2i)/n)(2/n) \\ &= 6\lim_{n\to\infty}(1/n)\left(-\textstyle\sum_{i=1}^{n} 1+(2/n)\sum_{i=1}^{n} i\right) \\ &= 6\lim_{n\to\infty}(1/n)\left(-n+(2/n)\tfrac{n(n+1)}{2}\right) \\ &= 6\lim_{n\to\infty}(-1)+\tfrac{n(n+1)}{n^2} \\ &= 6(-1+1) \\ &= 0 \end{aligned}$$

(2) Use the definition of the Riemann integral of two variables to calculate

$$\iint_R f\,dA$$

where $f(x,y)=xy$ and R is the rectangle in the xy plane with corners $(0,0)$, $(1,0)$, $(1,3)$, $(0,3)$. If you use lowerleft corners as sample points you will need the formula:

$$\sum_{k=1}^{n}(k-1)=\frac{(n-1)n}{2}.$$

Solution: Subdivide R into n^2 subrectangles each of area ΔA so that there are n subrectangles in the x direction and n in the y direction. Let R_{ij} denote the ith rectangle in the x direction and the jth rectangle in the y direction. Choose (x^*_{ij}, y^*ij) to be the lower left endpoint of each subrectangle. Then

$$\begin{aligned}
\Delta x &= 1/n \\
\Delta y &= 3/n \\
\Delta A &= 3/n^2 \\
x^*_{ij} &= (i-1)\Delta x = (i-1)/n \\
y^*_{ij} &= (j-1)\Delta y = 3(j-1)/n \\
f(x^*_{ij}, y^*_{ij}) &= 3(i-1)(j-1)/n^2
\end{aligned}$$

Consequently,

$$\begin{aligned}
\iint_R f\,dA &= \lim_{n\to\infty}\sum_{i=1}^{n}\sum_{j=1}^{n} f(x^*_{ij}, y^*_{ij})\Delta A \\
&= \lim_{n\to\infty}\sum_{i=1}^{n}\sum_{j=1}^{n} 9(i-1)(j-1)/n^4 \\
&= \lim_{n\to\infty}\frac{9}{n^4}\sum_{i=1}^{n}(i-1)\sum_{j=1}^{n}(j-1) \\
&= \lim_{n\to\infty}\frac{9}{n^4}\sum_{i=1}^{n}(i-1)\left(\frac{(n-1)n}{2}\right) \\
&= \lim_{n\to\infty}\frac{9}{n^4}\left(\frac{(n-1)n}{2}\right)\sum_{i=1}^{n}(i-1) \\
&= \lim_{n\to\infty}\frac{9}{n^4}\left(\frac{(n-1)n}{2}\right)\left(\frac{(n-1)n}{2}\right) \\
&= 9/4.
\end{aligned}$$

(3) Let $f(x,y) = x^3$ and let R be the square in the xy plane with corners $(-1,-1)$, $(-1,1)$, $(1,-1)$, and $(1,1)$. Without doing any calculations, explain why

$$\iint\limits_R f\,dA = 0.$$

Solution: Notice that $f(-x,y) = -f(x,y)$. For $x > 0$, the graph of $f(x,y)$ lies above the xy-plane in $\mathbb{R}^3$. For $x < 0$, the graph of $f(x,y)$ lies below the xy-plane. R is symmetric (in $\mathbb{R}^2$) about the y-axis. Since double integrals measure signed volume, $\iint\limits_R f\,dA$ is 0.

(4) Suppose that $f\colon \mathbb{R}^3 \to \mathbb{R}$ is a continuous function. Let R be a cube in $\mathbb{R}^3$. Modelling your answer on the definition of the Riemann integral in one and two variables, create a definition of

$$\iiint\limits_R f\,dV.$$

(Here V represents volume.)

Solution: Subdivide R into n smaller cubes each of volume ΔV. Number the cubes $C_1, \ldots, C_n$ and let (x_i^*, y_i^*, z_i^*) be a point in C_i. Then define

$$\iiint\limits_R f\,dV = \lim_{n\to\infty} f(x_i^*, y_i^*, z_i^*)\Delta V.$$

Problem B:

(1) Let $f\colon \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = e^{x^2}$.

(a) Find a series which represents f on the interval $[0,1]$.
Solution: Recall that

$$e^x = \sum_{i=0}^{\infty} \frac{x^i}{i!}$$

for all $x \in \mathbb{R}$. Thus,

$$e^{x^2} = \sum_{i=0}^{\infty} \frac{x^{2i}}{i!}$$

for all $x \in \mathbb{R}$.

(b) Use your answer from (a) that to obtain a series which equals

$$\int_0^1 f\,dx.$$

Solution:

$$\begin{aligned} \int_0^1 f\,dx &= \int_0^1 \sum_{i=0}^{\infty} \frac{x^{2i}}{i!}\,dx \\ &= \sum_{i=0}^{\infty} \int_0^1 \frac{x^{2i}}{i!}\,dx \\ &= \sum_{i=0}^{\infty} \frac{x^{2i+1}}{i!(2i+1)}\Big|_0^1 \\ &= \sum_{i=0}^{\infty} \frac{1}{i!(2i+1)}. \end{aligned}$$

(c) Use the first 10 terms of the series to estimate $\int_0^1 f\,dx$. You may wish to use Mathematica to do the computation.
Solution: According to *Mathematica*:

$$\sum_{i=0}^{9} \frac{1}{i!(2i+1)} = 1.462652.$$

(2) Let $f\colon \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = e^{x^2}$.

(a) Write $\int_0^1 f\,dx$ as the limit (as $n \to \infty$) of a Riemann sum.
Solution: Subdivide $[0,1]$ into n subintervals all of length Δx. In the ith subinterval choose a point x_i^*. Then

$$\int_0^1 f\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} e^{x_i^{*2}} \Delta x.$$

(b) Some of the pieces of the Riemann sum in (a) depend on n. Write each of them as an expression in terms of n.

Solution: $\Delta x = 1/n$ and, if we choose right hand endpoints, $x_i^* = i\Delta x$. Thus,

$$\int_0^1 f\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} e^{i^2/n^2}(1/n).$$

(c) Use $n = 10$ to estimate $\int_0^1 f\,dx$. You may wish to use Mathematica to help you with the computation.
Solution: With my choice of x_i^*, I get

$$\int_0^1 f\,dx \approx 1.55309$$

(d) Use Mathematica to evaluate numerically $\int_0^1 f\,dx$. Which of the answers from (2.c) or (1.c) is more accurate?
Solution: *Mathematica* says that $\int_0^1 f\,dx = 1.46265$. Thus the answer from (1c) is more accurate.

(3) Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x,y) = \sqrt{1+xy}$. Let R be the rectangle with corners $(0,1)$, $(0,2)$, $(3,1)$, $(3,2)$.

Solution Preliminaries: Both of the following parts will use this work. Suppose that R has been subdivided into n^2 subrectangles R_{ij} (n in the x direction and n in the y direction). The area of each subrectangle is then $\Delta A = 3/n^2$ since the area of R is 3. Choose a sample point (x_{ij}^*, y_{ij}^*) in R_{ij} to be the lower left corner of the subrectangle. Then:

$$\begin{aligned} x_{ij}^* &= (i-1)(3/n) \\ y_{ij}^* &= 1 + (j-1)(1/n) \end{aligned}$$

Then

$$\iint_R dA \approx \sum_{i=1}^{n} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\Delta A.$$

(a) Use a subdivision of R into 4 subrectangles to estimate $\iint_R f\,dA$. Be sure to specify the points $\mathbf{x}_{ij}^*$. You should probably use Mathematica to help with the computation.
Solution: Plugging $n = 2$ into the equation above and using *Mathematica* we obtain 4.03794.

(b) Use a subdivision of R into 16 subrectangles to estimate $\iint_R f\,dA$. Be sure to specify the points $\mathbf{x}_{ij}^*$.

Solution: Plugging $n = 4$ into the equation above and using *Mathematica* we obtain 4.63982. It may interest you to compare this to *Mathematica*'s approximation to the integral: 5.27674.

Problem C: In the 5th edition of the text, do problems 1, 3, 7, 9, 11, 14, 15, 16, 17, 23 on page 840. The answers to the odd numbered problems are in the back.

See book or answer key

Problem D: Let $f(x,y) = -(x^2+y^2)+4$. The graph of f intersects the xy plane in a circle of radius 2 centered at the origin. Let S be the region between the xy plane and the graph of f.

(1) Set up an iterated integral in Cartesian coordinates representing the volume of S. Use Mathematica to solve the integral.

Solution:

$$\iint_S f\,dA = \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} f(x,y)\,dy\,dx = \int_{-2}^{2}\int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} f(x,y)\,dx\,dy.$$

Mathematica gives an answer of 8π.

(2) Set up an iterated integral in polar coordinates representing the volume of S. Solve it without using Mathematica to find the volume of S.

Solution: In polar coordinates, $\widehat{f}(r,\theta) = -r^2+4$ and $\widehat{S}$ is the region $0 \le r \le 2$ and $0 \le \theta \le 2\pi$. Then:

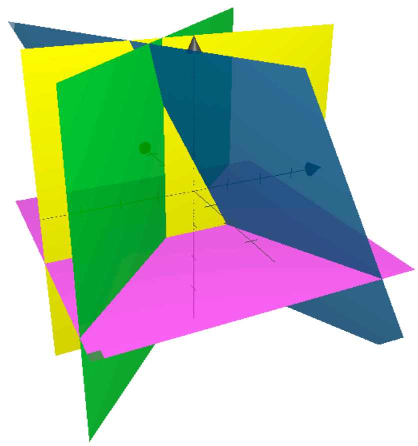
$$\iint_S f\,dA = \iint_{\widehat{S}} \widehat{f}\,r\,d\widehat{A} = \int_0^{2\pi}\int_0^2 (-r^2+4)r\,dr\,d\theta.$$

We can easily calculate this integral:

$$\begin{aligned} \textstyle\int_0^{2\pi}\int_0^2 (-r^2+4)r\,dr\,d\theta &= \textstyle\int_0^{2\pi}\int_0^2 -r^3+4r\,dr\,d\theta \\ &= \textstyle\int_0^{2\pi}(-16/4+2(2)^2)d\theta \\ &= 8\pi. \end{aligned}$$

Problem E: Find the mass of the pyramid with base in the plane $z = -6$ and sides formed by the three planes $y = 0$ and $y - x = 4$ and $2x + y + z = 4$ if the density of the solid is given by $\delta(x, y, z) = y$. (The mass of a solid P with density function δ is $\iiint\limits_P \delta \, dV$.

Solution: Below is the graph of the 4 planes:



A good strategy for this problem is to write down a triple integral $\iiint\limits_P 1\, dV$ representing the volume of P. We can then replace the "1" with δ.

Let's begin by choosing which direction to slice the pyramid. This problem will work out no matter which direction we slice it, so let's just pick z-slices. Then the volume of P is

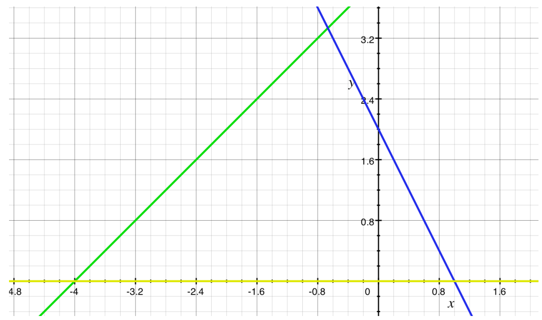
$$\operatorname{vol} P = \int_{z_{\min}}^{z_{\max}} A(z)\, dz$$

where $A(z)$ is the area of the z-slices.

Let's figure out $z_{\min}$ and $z_{\max}$. Clearly, $z_{\min} = -6$. The value of $z_{\max}$ occurs when we are at the top of the pyramid. That is the point where the planes $y = 0$, $y - x = 4$, and $2x + y + z = 4$ intersect. A little algebra shows that $z_{\max} = 12$. Thus,

$$\operatorname{vol} P = \int_{-6}^{12} A(z)\, dz.$$

Now we must figure out a formula for $A(z)$. Below is a graph of a typical z-slice in the $x - y$ plane. The colors of the lines have been chosen to coincide with the colors of the planes above.



$A(z)$ is the area of the triangle bounded by the three planes. To compute $A(z)$, let's take y-slices so that

$$A(z) = \int_{y_{\min}}^{y_{\max}} L(y)\, dy$$

where $L(y)$ is the length of the y–slice. Clearly, $y_{\min} = 0$. The value of $y_{\max}$ occurs when we are taking the y-slice at the very top of the triangle. The top of the triangle is where the lines $x+y=4$ and $2x+y+z=4$ intersect (in our z–slice). Solving for x the equations of these lines are:

$$\begin{aligned} x &= y-4 \\ x &= \tfrac{1}{2}\left(-y+4-z\right) \end{aligned}$$

Equating these and solving for y, we find that $y_{\max} = 4 - z/3$. Thus,

$$A(z) = \int_{0}^{4-z/3} L(y)\, dy.$$

Notice also that the length of a y-slice is

$$L(y) = (-y+4-z)/2 - (y-4) = \int_{y-4}^{(-y+4-z)/2} 1\, dx$$

. That is, for fixed y and z, $y-4 \le x \le (-y+4-z)/2$.. Consequently,

$$\operatorname{vol} P = \int_{-6}^{12} \int_{0}^{4-z/3} \int_{y-4}^{(-y+4-z)/2} 1\, dx\, dy\, dz.$$

Thus,

$$\operatorname{density} P = \int_{-6}^{12} \int_{0}^{4-z/3} \int_{y-4}^{(-y+4-z)/2} \delta(x,y,z)\, dx\, dy\, dz.$$

Evaluating by hand or plugging into *Mathematica* we get:

$$\operatorname{density} P = 243.$$

Problem F: In the 5th edition of the text, do problems 1 and 3 on pages 850-851 and problems 19 and 22 on page 870. (pg 851 # 24 and pg 870 #21 are extra-credit)

Here is a solution for Problem 22 on page 870.

We have:

$$\begin{array}{rcl} s & = & x-y \\ t & = & x+y \end{array}$$

as our coordinate change. Some algebra shows that:

$$\begin{array}{rcl} x & = & s/2+t/2 \\ y & = & -s/2+t/2 \end{array}$$

Thus, the Jacobian of the coordinate change from x,y coordinates to s,t coordinates is $\begin{pmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{pmatrix}$. This has determinant $1/2$.

The given region R is $x^2+y^2 \leq 1$. Equivalently, $(s/2+t/2)^2+(-s/2+t/2)^2 \leq 1$. Some algebra produces $s^2+t^2 \leq 2$. Thus, the transformed region $\widehat{R}$ is the disc centered at the origin of the s,t plane of radius $\sqrt{2}$. We have:

$$\iint_R f\,dA = \iint_{\widehat{R}} \widehat{f}(1/2)\,d\widehat{A}.$$

Since $f(x,y) = \sin(x+y)$, plugging in for x and y, we obtain $\widehat{f}(s,t) = \sin(t)$. Consequently:

$$\iint_R f\,dA = \int_{-\sqrt{2}}^{\sqrt{2}} \int_{-\sqrt{2-t^2}}^{\sqrt{2-t^2}} \sin(t)/2\,ds\,dt$$

This can be evaluated using *Mathematica* to find that the answer is 0. We could also have spotted this in the original problem by noticing that if $f(x,y) = \sin(x+y)$ then $f(-x,-y) = -f(x,y)$. So rotating a point (x,y) on the xy- plane by 180° changes the sign of $f(x,y)$. Since the region R is symmetric under 180° rotation, and since the double integral measures signed volume, the double integral must be 0.