Study Guide/Practice Exam 3

This study guide/practice exam covers only the material since exam 2. The final exam, however, is cumulative so you should be sure to thoroughly study earlier material. The distribution of content on this practice exam is not necessarily representative of the distribution of content on the actual exam.

(1) Suppose that f is a function which is the solution to the initial value problem

$$\begin{aligned} \tfrac{\partial}{\partial x} f(x,y) + \tfrac{\partial}{\partial y} f(x,y) &= x f(x,y) \\ f(0,0) &= 1 \end{aligned}$$

A theorem from differential equations guarantees that $f \in C^2(\mathbb{R}^2)$.

(a) Explain what $f \in C^2(\mathbb{R}^2)$ means.
(b) State a theorem which guarantees that $\frac{\partial}{\partial x \partial y} f(x,y) = \frac{\partial}{\partial y \partial x} f(x,y)$ for all $(x,y) \in \mathbb{R}^2$.

Solution: That $f \in C^2(\mathbb{R}^2)$ means that f, all first partial derivatives of f, and all second partial derivatives of f are continuous at every point in $\mathbb{R}^2$. Clairut's theorem says that if $f \in C^2(D)$ where D is a disc of positive radius in $\mathbb{R}^2$ containing a point $\mathbf{a}$ then the mixed partial derivatives of f at a are equal. Thus, if $f \in C^2(\mathbb{R}^2)$, then the mixed partials of f are equal at every point in $\mathbb{R}^2$.

(2) Let $f(x,y) = x^2 - \sin(x) + \cos(y)$. Find the first and second MacLaurin polynomials for f.

Solution: The first Taylor polynomial is $P_1(x,y) = 1 - x$. The second is $P_2(x,y) = 1 - x + x^2 - y^2$.

(3) Let $\phi(t) = (t^2, t, \sin(t))$. Let $f(x,y,z) = x - y + z$. Use the chain rule to find $\frac{d}{dt} f \circ \phi(t)$.

Solution:

$$\begin{aligned} \phi'(t) &= (2t, 1, \cos(t)) \\ \nabla f(x,y,z) &= (1,-1,1) \\ \tfrac{d}{dt} f \circ \phi(t) &= (1,-1,1) \cdot (2t, 1, \cos(t)) \end{aligned}$$

Thus, $\frac{d}{dt} f \circ \phi(t) = 2t - 1 + \cos(t)$.

(4) Let $\phi(t) = (\cos t, \cos t - \sin t)$. Let L be the graph in $\mathbb{R}^2$ of the line $y - 2x = 4$. Find points $t_{\min}, t_{\max} \in [0, 2\pi)$ so that $\phi(t_{\min})$ is the closest point on the ellipse to L and $\phi(t_{\max})$ is the farthest point. You do not need to determine which is which.

Solution: The distance from (x, y) in $\mathbb{R}^2$ to the line defined by the equation $y - 2x = 0$ is

$$\mathrm{dist}(x, y) = \frac{|(x, y) \cdot (-2, 1)|}{\sqrt{5}}$$

The line L does not go through the origin, but if we shift everything down by 4 we will get $y - 2x = 0$. Shifting everything down by 4 does not change the distance between objects. Thus, the distance from (x, y) to L is the same as $\mathrm{dist}(x, y - 4)$. To minimize the distance we will minimize the square of the distance function. Let

$$s(x, y) = \frac{(-2x + y - 4)^2}{5}$$

We wish to minimize $s \circ \phi(t)$. We begin by finding the critical points using the chain rule.

$$\begin{aligned} \nabla s(x, y) &= \tfrac{2}{5}(-2x + y - 4)(-2, 1) \\ \phi'(t) &= (-\sin t, -\sin t + \cos t) \\ \tfrac{d}{dt} s \circ \phi(t) &= \tfrac{2}{5}(-\cos t - \sin t - 4)(\sin t + \cos t) \end{aligned}$$

Set this last derivative equal to zero and solve for t. To do so, notice that $(-\cos t - \sin t - 4)$ is never 0. Thus, the derivative is equal to zero only if $\sin t = -\cos t$. That is, if $\tan t = -1$. For $t \in [0, 2\pi)$, $\tan t = -1$ only if $t = 3\pi/4$ or $t = 7\pi/4$. These must be the t values creating the maximum and minimum distances to the line L.

(5) Let $f(x, y) = x^3 + x^2 y - y^2$. Find and classify all critical points of f.

Solution: We begin by calculating:

$$\nabla f(x, y) = (3x^2 + 2xy, x^2 - 2y)$$

and

$$Hf(x, y) = \begin{pmatrix} 6x + 2y & 2x \\ 2x & -2 \end{pmatrix}$$

We have $\nabla f(x, y) = (0, 0)$ if $(x, y) = (0, 0)$ or if $(x, y) = (3, 9/2)$. These are our critical points. Plugging them into $Hf(x, y)$ and using the second derivative test we discover that $(0, 0)$ is a degenerate critical point and that $(3, 9/2)$ is a saddle point.

(6) Let $f(x, y) = e^{x^2+y^2} - e^{-(x^2+y^2)}$. Find and classify all critical points of f.

Solution:

$$\begin{aligned} f_x(x,y) &= 2x(e^{x^2+y^2} + e^{-(x^2+y^2)} \\ f_y(x,y) &= 2y(e^{x^2+y^2} + e^{-(x^2+y^2)} \end{aligned}$$

The only critical point of f is, therefore, $(0,0)$. Then

$$\begin{aligned} f_{xx}(x,y) &= 2(e^{x^2+y^2} + e^{-(x^2+y^2)}) + 4x^2(\text{ something }) \\ f_{yy}(x,y) &= 2(e^{x^2+y^2} + e^{-(x^2+y^2)} + 4y^2(\text{ something }) \\ f_{yx}(x,y) &= 2x(\text{ something }) \end{aligned}$$

Consequently, $Hf(0,0) = \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}$. By the second derivative test, $(0,0)$ is a minimum of f.

(7) Let $g(x,y) = \frac{1}{xy}$. Find the points on the graph of g which are closest to the origin in $\mathbb{R}^3$. (Hint: Let $s(x,y,z) = x^2+y^2+z^2$ be the square of the distance from (x,y,z) to the origin and set $z = g(x,y)$.)

Solution: Let $s(x,y) = x^2 + y^2 + \frac{1}{x^2y^2}$. We have

$$\nabla s(x,y) = (2x - \frac{2}{y^2x^3}, 2y - \frac{2}{y^3x^2}).$$

Setting $\nabla s(x,y) = (0,0)$ and solving we get four critical points $(1,1)$, $(-1,1)$, $(1,-1)$, and $(-1,-1)$. Call these A, B, C, and D respectively.

Then $Hs(x,y) = \begin{pmatrix} 2+6y^{-2}x^{-4} & 4y^{-3}x^{-3} \\ 4y^{-3}x^{-3} & 2+6y^{-4}x^{-2} \end{pmatrix}$. Consequently:

$$\begin{aligned} Hs(A) &= \begin{pmatrix} 8 & 4 \\ 4 & 8 \end{pmatrix} & \qquad Hs(B) &= \begin{pmatrix} 8 & -4 \\ -4 & 8 \end{pmatrix} \\ Hs(C) &= \begin{pmatrix} 8 & -4 \\ -4 & 8 \end{pmatrix} & \qquad Hs(D) &= \begin{pmatrix} 8 & 4 \\ 4 & 8 \end{pmatrix} \end{aligned}$$

By the second derivative test, these are all minima.

(8) A company operates two plants which manufacture the same item and whose total cost functions are

$$\begin{aligned} C_1 &= 8.5 + 0.03q_1^2 \\ C_2 &= 5.2 + 0.04q_2^2 \end{aligned}$$

where q_1 and q_2 are the quantities produced by each plant. If the item costs p dollars then

$$p = 60 - .04(q_1 + q_2).$$

The goal is to find values for q_1 and q_2 which will maximize the company's profit.

Carefully set up and describe how you would solve this problem using multi-variable derivatives. You need not actually perform the calculations.

Solution: Let $P(q_1, q_2) = (q_1+q_2)p-(C_1+C_2)$. This is a function of q_1 and q_2 which represents the profit of the company. I would plug in the equations for p, C_1, and C_2 and would find the critical points and then use the second derivative test to determine which critical points were maxima of P.

(9) Let $f(x, y) = x + y$ and let R be the rectangle in $\mathbb{R}^2$ with corners $(1, 1)$, $(1, 3)$, $(2, 1)$ and $(2, 3)$.

(a) Subdivide R into four subrectangles of equal area and write down a sum which approximates $\iint\limits_R f\,dA$.

Solution: Since the area of R is 2, the area of each subrectangle is $\Delta A = 2/4 = 1/2$. Choose lower left corners for sample points (for example). Then we have sample points $(1, 1)$, $(3/2, 1)$, $(2, 1)$, and $(3/2, 2)$. The following sum approximates the double integral:

$$\Delta A(f(1,1)+f(3/2,1)+f(2,1)+f(3/2,2)) \quad = (1/2)(1+1+3/2+1+2+1+3/2+2)$$

(b) Use lower left corners of subdivisions of R and the limit definition of the Riemann integral to calculate $\iint\limits_R f\,dA$. You will need the formula;

$$\sum_{k=0}^{n}(k-1) = \frac{(n-1)n}{2}.$$

Solution: Subdividing R into n^2 rectangles we have $\Delta A = 2/n^2$. Let R_{ij} be the ith subrectangle in the x direction and the jth subrectangle in the y direction. Then with lower left corners of sample points (x^*_{ij}, y^*_{ij}) we have

$$\begin{aligned} x^*_{ij} &= 1+(i-1)/n \\ y^*_{ij} &= 1+2(j-1)/n \\ f(x^*_{ij}, y^*_{ij}) &= 2+(i-1)/n+2(j-1)/n \end{aligned}$$

Thus,

$$
\begin{aligned}
\iint_R f\,dA &= \lim_{n\to\infty} \sum_{i=1}^{n}\sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\Delta A \\
&= \lim_{n\to\infty} \tfrac{2}{n^2} \sum_{i=1}^{n}\sum_{j=1}^{n} 2 + (i-1)/n + 2(j-1)/n \\
&= \lim_{n\to\infty} \tfrac{2}{n^2} \sum_{i=1}^{n} \left(\sum_{j=1}^{n} 2 + \sum_{j=1}^{n} (i-1)/n + \sum_{j=1}^{n} 2(j-1)/n \right) \\
&= \lim_{n\to\infty} \tfrac{2}{n^2} \sum_{i=1}^{n} \Big(2n + (i-1) + (n-1) \Big) \\
&= \lim_{n\to\infty} \tfrac{2}{n^2} \left(\sum_{i=1}^{n} 2n + \sum_{i=1}^{n} (i-1) + \sum_{i=1}^{n} (n-1) \right) \\
&= \lim_{n\to\infty} \tfrac{2}{n^2} \Big(2n^2 + n(n-1)/2 + (n-1)n \Big) \\
&= 4 + 1 + 2 \\
&= 7.
\end{aligned}
$$

(c) Write $\iint_R f\,dA$ as an iterated integral and solve.

Solution:

$$
\begin{aligned}
\iint_R f\,dA &= \int_1^3 \int_1^2 (x+y)\,dx\,dy \\
&= \int_1^3 (x^2/2 + yx)\Big|_1^2\, dy \\
&= \int_1^3 (3/2) + y\,dy \\
&= 7.
\end{aligned}
$$

(10) For the following functions f and regions R set up (but do not solve) an iterated integral equal to $\iint_R f\,dA$. Your answer should be something that can be plugged into *Mathematica* to find the answer.

(a) $f(x, y) = x^3 y$ and R is a disc of radius 1 centered at the point $(1, -1)$.

Solution:

$$
\iint_R f\,dA = \int_{-} 1^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} x^3 y\,dy\,dx
$$

(b) $f(x, y) = \sin(xy)$ and R is the triangular region with corners $(0, 0)$, $(2, 0)$, and $(1, 5)$.

Solution:

$$\iint\limits_R f\,dA = \int_0^5 \int_{y/5}^{-y/5+2} \sin(xy)\,dx\,dy$$

(c) $f(x,y) = x^2 - y^2$ and R is the region bounded by the graphs of $y = x^5$ and $y = x^3$.

Solution:

$$\iint\limits_R f\,dA = \int_{-1}^0 \int_{x^3}^{x^5} x^2 - y^2\,dy\,dx + \int_0^1 \int_{x^5}^{x^3} x^2 - y^2\,dy\,dx$$

(11) Set up iterated triple integrals to find the volumes of the following objects in $\mathbb{R}^3$. You do not need to solve the integrals.

(a) The region trapped between the graphs of $y = -1$, $y = 1$, $y = x^3$, $z = x$, and $z = 5$.

Solution: We use y-slices so that $\iiint\limits_R 1\,dV = \int_{-} 1^1 A(y)\,dy$, where $A(y)$ is the area of a y slice. Figure 1 shows a typical y-slice.

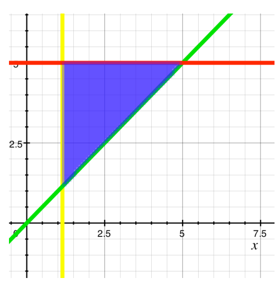

FIGURE 1. The colors of the lines have been chosen to correspond to the colors of the 3D object in the Practice Exam. The yellow line has the equation $x = \sqrt[3]{y}$. The green line has equation $z = x$, and the red line has the equation $z = 5$.

Examining the graph we see that $A(y) = \int_{\sqrt[3]{y}}^5 \int_x^5 1\,dz\,dx$. Thus,

$$\iiint\limits_R 1\,dV = \int_{-} 1^1 A(y)\,dy = \int_{-1}^1 \int_{\sqrt[3]{x}}^5 \int_x^5 1\,dz\,dx\,dy.$$

(b) The region which is trapped between the cylinders $x^2 + y^2 = 1$ and $x^2 + z^2 = 1$.

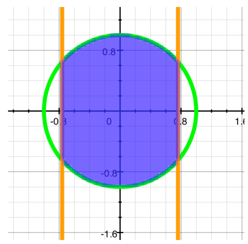

FIGURE 2. The green circle is $x^2 + y^2 = 1$ and the orange lines are is $x^2 + z^2 = 1$.

Solution: We choose to use z–slices. In Figure 2 is drawn a typical z-slice. The green circle is $x^2 + y^2 = 1$. The left orange line is $x = -\sqrt{1-z^2}$ and the right orange line is $\sqrt{1-z^2}$. Since $-1 \leq z \leq 1$, we have $\iiint\limits_R 1\,dV = \int_{-1}^{1} A(v)\,dV$ where $A(v)$ is the area of a z slice. By examining the graph of the z-slice we see that $A(z) = \int_{-\sqrt{1-z^2}}^{\sqrt{1-z^2}} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} 1\,dy\,dx$. Thus,

$$\iiint\limits_R 1\,dV = \int_{-1}^{1} \int_{-\sqrt{1-z^2}}^{\sqrt{1-z^2}} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} 1\,dy\,dx\,dz.$$

(12) Carefully state and explain the Change of Variables Theorem for double integrals.

Solution: Suppose that $f\colon \mathbb{R}^2 \to \mathbb{R}$ is continuous and that $x = x(u,v)$ and $y = y(u,v)$ are a differentiable change of coordinates. Let $R \subset \mathbb{R}^2$ be a region in the xy plane for which $\iint_R f\,dA$ exists. Let $\widehat{R}$ be the corresponding region in the u, v plane. Let $\hat{f}$ be the composition of f with the coordinate change. Let $J = \begin{pmatrix} x_u & x_v \\ y_u & y_v \end{pmatrix}$. Then

$$\iint\limits_R f\,dA = \iint\limits_{\widehat{R}} \hat{f}\,|\det J|\,d\widehat{A}.$$

The theorem gives a way of converting an integral of f over a region in the xy plane to an integral over a region in the uv plane. The determinant of the Jacobian measures the infinitessimal change in area due to the coordinate change. It is necessary to include this so that the Riemann sum representing the integral on the left is equal to the Riemann sum representing the integral on the right.

(13) Let $f(x,y) = y\sqrt{4-(x^2+y^2)}$. Let R be the half disc defined by $x^2+y^2 \leq 1$ and $y \geq 0$. Set up an iterated integral in polar coordinates which is equal to $\iint\limits_R f\,dA$.

Solution: In polar coordinates, we have $y = r\sin\theta$ and $r^2 = x^2 + y^2$. Thus,

$$\hat{f}(r,\theta) = r\sin\theta\sqrt{4-r^2}.$$

The half disc can be parameterized as $0 \leq r \leq 1$ and $0 \leq \theta \leq \pi$. Thus

$$\iint\limits_R f\,dA = \int_0^\pi \int_0^1 4r^2 \sin\theta\sqrt{4-r^2}\,dr\,d\theta.$$

(14) The following problems each give a function f, a region R, and a change of coordinates. For each, write down an iterated integral in the new coordinate system which equals $\iint\limits_R f\,dA$.

(a) $f(x,y) = x - y$. R is the triangle with corners $(0,0)$, $(3,1)$ and $(0,2)$. The coordinate change is given by $x = s - t$, $y = -s - t$.

Solution: We have $\hat{f}(s,t) = 2s$ and $\det J = -2$. Since the coordinate change is a linear coordinate change lines are taken to lines. Thus, we need only compute the image of the three corners of R in the st plane. To do this we solve for s and t in terms of x and y, obtaining:

$$\begin{aligned} s &= x/2 - y/2 \\ t &= -x/2 - y/2 \end{aligned}$$

Thus $\widehat{R}$ is a triangular region with vertices $(0,0)$, $(1,-2)$, and $(-1,-1)$. Thus, we have

$$\begin{aligned} \iint\limits_R f\,dA &= \iint\limits_{\widehat{R}} 4s\,d\widehat{A} \\ &= \int_{-1}^{1}\int_{-(s+1)/2-1}^{s} 4s\,dt\,ds + \int_0^1\int_{-(s+1)/2-1}^{-2s} 4s\,dt\,ds \end{aligned}$$

(b) $f(x,y) = \sqrt[4]{x+y}$. R is the region bounded by $x^2 - y^2 = 1$, $x^2 - y^2 = 4$, $y = 0$, and $y = x/2$. The region appears in Figure 3. The coordinate change is given by $x = r\cosh\theta$, $y = r\sinh\theta$. It may help to remember the following facts:

$$\begin{aligned} \tfrac{d}{d\theta}\cosh\theta &= \sinh\theta \\ \tfrac{d}{d\theta}\sinh\theta &= \cosh\theta \\ \cosh^2\theta - \sinh^2\theta &= 1 \\ \cosh\theta &= (e^\theta + e^{-\theta})/2 \\ \sinh\theta &= (e^\theta - e^{-\theta})/2 \end{aligned}$$

Solution: The new function is

$$\begin{aligned} \hat{f}(r,\theta) &= (r\cosh\theta + r\sinh\theta)^{1/4} \\ &= r^{1/4}e^{\theta/4} \end{aligned}$$

The new region is defined bounded by

$$\begin{aligned} r^2 &= 1 \\ r^2 &= 4 \\ r\sinh\theta &= 0 \\ 2r\sinh\theta &= r\cosh\theta \end{aligned}$$

The hyperbolas $x^2 - y^2 = 1$ and $x^2 - y^2 = 4$ each have two branches, if $f > 0$ we are on the right branches which is what we want. Thus, two of the curves bounding our new region $\widehat{R}$ are

$$\begin{aligned} r &= 1 \\ r &= 2 \end{aligned}$$

Thus, $r > 0$ for all the points in $\widehat{R}$. Consequently, our other lines bounding $\widehat{R}$ must be:

$$\begin{aligned} \sinh\theta &= 0 \\ 2(e^\theta - e^{-\theta}) &= e^\theta + e^{-\theta} \end{aligned}$$

The first of these happens only if $\theta = 0$ and the second only if $\theta = \ln(3)/2$. Thus our region $\widehat{R}$ is the rectangle bounded by

$$\begin{aligned} r &= 1 \\ r &= 2 \\ \theta &= 0 \\ \theta &= (\ln 3)/2 \end{aligned}$$

The Jacobian of the coordinate change is:

$$J = \begin{pmatrix} \cosh\theta & r\sinh\theta \\ \sinh\theta & r\cosh\theta \end{pmatrix}$$

It has determinant:

$$\det J = r\cosh^2\theta - r\sinh^2\theta = r$$

Hence,

$$\begin{aligned} \iint\limits_R f\,dA &= \iint\limits_{\widehat{R}} \hat{f} r\,d\widehat{A} \\ &= \int_1^2 \int_0^{(ln3)/2} r^{5/4} e^{\theta/4}\,d\theta\,dr \end{aligned}$$

(c) Let $f(x, y) = 1$. Let R be the elliptical region $Ax^2 + Bxy + Cy^2 \leq 1$, where A, B, and C are positive constants such that $C > B^2/(4A^2)$. Use the coordinate change

$$\begin{aligned} s &= \left(x + \tfrac{B}{2A}y\right)\sqrt{A} \\ t &= y\sqrt{C - \tfrac{B^2}{4A}}. \end{aligned}$$

Computing $\iint\limits_R f\,dA$ will give the area of R.

Solution: For convenience we let $k = \sqrt{C - B^2/(4A)}$. The coordinate change can be rewritten as:

$$\begin{aligned} x &= \tfrac{s}{\sqrt{A}} - \tfrac{Bt}{2Ak} \\ y &= t/k \end{aligned}$$

Thus, the determinant of the Jacobian is $\det J = \frac{1}{k\sqrt{A}}$. Our region is determined as follows:

$$\begin{aligned} Ax^2 + Bxy + Cy^2 &\leq 1 \\ A(\tfrac{s}{\sqrt{A}} - \tfrac{Bt}{2aK})^2 + B(\tfrac{s}{\sqrt{A}} - \tfrac{Bt}{2Ak})(\tfrac{t}{k}) + \tfrac{Ct^2}{k^2} &\leq 1 \\ A\left(\tfrac{s^2}{A} - \tfrac{Bts}{\sqrt{A}Ak} + \tfrac{B^2t^2}{4A^2k^2}\right) + \tfrac{Bst}{\sqrt{A}k} - \tfrac{B^2t^2}{2Ak^2} + \tfrac{Ct^2}{k^2} &\leq 1 \\ s^2 - \tfrac{Bts}{\sqrt{A}k} + \tfrac{B^2t^2}{4Ak^2} + \tfrac{Bst}{\sqrt{A}k} - \tfrac{B^2t^2}{2Ak^2} + \tfrac{Ct^2}{k^2} &\leq 1 \\ k^2s^2 + (\tfrac{B^2}{4A} - \tfrac{B^2}{2A} + C)t^2 &\leq k^2 \\ k^2s^2 + (-\tfrac{B^2}{4A} + C)t^2 &\leq k^2 \\ k^2s^2 + k^2t^2 &\leq k^2 \\ s^2 + t^2 &\leq 1 \end{aligned}$$

Thus the new region $\widehat{R}$ is the disc of unit radius in the st plane. Consequently,

$$\begin{aligned} \iint\limits_R f\,dA &= \iint\limits_{\widehat{R}} f(\tfrac{1}{k\sqrt{A}})\,d\widehat{A} \\ &= \tfrac{1}{k\sqrt{A}} \iint\limits_{\widehat{R}} 1\,d\widehat{A} \\ &= \tfrac{1}{k\sqrt{A}}\pi \\ &= \tfrac{2\pi}{\sqrt{4AC-B^2}} \end{aligned}$$