MA 122: Chain Rule Example

Problem: Let $\phi(t) = (t, t^2 - 5, -t^2 + 1)$. Let P be the plane defined by the equation $x + y + z = 0$. Find a number t_0 such that $\phi(t_0)$ minimizes the distance between the image of ϕ and the plane P.

Solution: Let dist($\mathbf{v}$) denote the distance from $\mathbf{v}$ to P. Then for some number t_0:

$$\text{dist}(\phi(t_0)) \leq \text{dist}(\phi(t)) \Leftrightarrow \text{dist}^2(\phi(t_0)) \leq \text{dist}^2(\phi(t)).$$

That is, the minimum of dist occurs at the same t value as the minimum of dist^2. We choose to find a t_0 such that the square of the distance is minimized.

Let

$$f(x,y,z) = \left(\frac{|(x,y,z)\cdot(1,1,1)|}{\sqrt{3}}\right)^2 = \frac{(x+y+z)^2}{3}.$$

Recall that $f(x,y,z)$ is the square of the distance from a point $(x,y,z) \in \mathbb{R}^3$ to the plane P since $(1,1,1)$ is a normal vector for P. We wish to minimize the function $f \circ \phi(t)$. We will do this by setting $\frac{d}{dt} f \circ \phi(t)$ equal to zero and solving for t.

The chain rule says that

$$\frac{d}{dt} f \circ \phi(t) = \nabla f(\phi(t)) \cdot \phi'(t).$$

Calculations show that:

$$\nabla f(x,y,z) = \frac{2}{3}(x+y+z)(1,1,1)$$

and

$$\phi'(t) = (1, 2t, -2t).$$

Thus,

$$\begin{aligned} \tfrac{d}{dt} f \circ \phi(t) &= \tfrac{2}{3}(x+y+z)(1,1,1)\cdot(1,2t,-2t) \\ &= \tfrac{2}{3}(t + t^2 - 5 - t^2 + 1)(1 + 2t - 2t) \\ &= \tfrac{2}{3}(t-4). \end{aligned}$$

This is equal to zero if and only if $t = 4$.

The second derivative of $f \circ \phi(t) = 1$ so $t_0 = 4$ is the minimum value of $f \circ \phi(t)$.